

Rank four symplectic bundles without theta divisors over a curve of genus two

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Abstract

The moduli space \mathcal{M}_2 of rank four semistable symplectic vector bundles over a curve X of genus two is an irreducible projective variety of dimension ten. Its Picard group is generated by the determinantal line bundle Ξ . The base locus of the linear system $|\Xi|$ consists of precisely those bundles without theta divisors, that is, admitting nonzero maps from every line bundle of degree -1 over X . We show that this base locus consists of six distinct points, which are in canonical bijection with the Weierstrass points of the curve. We relate our construction of these bundles to another of Raynaud and Beauville using Fourier–Mukai transforms. As an application, we prove that the map sending a symplectic vector bundle to its theta divisor is a surjective map from \mathcal{M}_2 to the space of even 4Θ divisors on the Jacobian variety of the curve.

1 Introduction

In this section we introduce the objects we will be studying, and give a summary of the paper. Let X be a complex projective curve which is smooth and irreducible of genus $g \geq 2$.

Definition: A vector bundle $W \rightarrow X$ is *symplectic* (resp., *orthogonal*) if there is a bilinear nondegenerate antisymmetric (resp., symmetric) form ω on $W \times W$ with values in a line bundle L , which for us will often be the trivial bundle \mathcal{O}_X . Note that a symplectic bundle is necessarily of even rank.

Let r and d be integers with $r \geq 1$. The moduli space of semistable vector bundles of rank r and degree d over X is denoted $\mathcal{U}(r, d)$. For any line

bundle $L \rightarrow X$ of degree d , the closed subvariety of $\mathcal{U}(r, d)$ of bundles with determinant L is denoted $\mathcal{SU}(r, L)$. References for these objects include Seshadri [35] and Le Potier [21]. The variety $\mathcal{U}(1, d)$ is the d th Jacobian variety of X , and will be denoted J^d ; see for example Birkenhake–Lange [9], Chap. 11, for details. The main object of interest for us is the moduli space of semistable symplectic vector bundles of rank $2n$ over X , which is denoted \mathcal{M}_n . We have

Theorem 1 *\mathcal{M}_n is canonically isomorphic to $\mathcal{M}(\mathrm{Sp}_n \mathbb{C})$, the moduli space of semistable principal $\mathrm{Sp}_n \mathbb{C}$ -bundles over X , and the natural map $\mathcal{M}_n \rightarrow \mathcal{SU}(2n, \mathcal{O}_X)$ is injective.*

Proof

See [11], chap. 1, or [13] for a sketch.

This allows us to use Ramanathan’s results in [30] and [31] (especially Theorem 5.9) to give information about \mathcal{M}_n . We find that \mathcal{M}_n is an irreducible, normal, projective variety of dimension $n(2n + 1)(g - 1)$.

Generalised theta divisors and the theta map

For a semistable vector bundle $W \rightarrow X$ of rank r and trivial determinant, we consider the set

$$S(W) := \{L \in J^{g-1} : h^0(X, L \otimes W) > 0\}.$$

If $S(W) \neq J^{g-1}$ then it is the support of a divisor $D(W)$ on J^{g-1} linearly equivalent to $r\Theta$, called the *theta divisor* of W . It is not hard to show that this only depends on the S -equivalence class of W (see for example [11], chap. 6). The association $D \mapsto D(W)$ defines a rational map $\mathcal{SU}_X(r, \mathcal{O}_X) \dashrightarrow |r\Theta|$.

Henceforth we suppose that $r = 2n$ and consider the map

$$D : \mathcal{M}_n \dashrightarrow |2n\Theta|.$$

We can be more precise about the image of D . Recall that the Serre duality involution $\iota : J^{g-1} \rightarrow J^{g-1}$ is given by $L \mapsto K_X L^{-1}$. Since $\iota^* \Theta = \Theta$, we have induced involutions ι^* on $H^0(J^{g-1}, 2n\Theta)$. The projectivisations of the $+1$ and -1 eigenspaces of this involution correspond to the spaces of ι^* -invariant divisors and are denoted $|2n\Theta|_+$ and $|2n\Theta|_-$ respectively. We have $h^0(J^{g-1}, 2n\Theta)_\pm = 2n^g \pm 2^{g-1}$.

Lemma 2 *The image of $D : \mathcal{M}_n \dashrightarrow |2n\Theta|$ is contained in $|2n\Theta|_+$.*

Proof

See Beauville [6], section 2.

We denote Ξ the line bundle $D^*O(1)$. We claim that the base locus of $|\Xi|$ is exactly the set of bundles W with $S(W) = J^{g-1}$. For any $L \in J^{g-1}$, we write H_L for the hyperplane of divisors containing L , that is, the image of L under the standard map $\phi_{2n\Theta_+} : J^{g-1} \rightarrow |2n\Theta|_+^*$. Then

$$\begin{aligned}
S(W) = J^{g-1} &\iff W \in \bigcap_{L \in J^{g-1}} \text{Supp}(D^*H_L), \text{ by definition} \\
&\iff W \in \bigcap_{H \in |2n\Theta|_+} \text{Supp}(D^*H), \\
&\quad \text{since the image of } \phi_{2n\Theta_+} \text{ is nondegenerate} \\
&\iff W \text{ belongs to every } \Xi \text{ divisor, as } |2n\Theta|_+ = |\Xi|^* \\
&\iff W \text{ is a base point of } |\Xi|.
\end{aligned}$$

Several results on such bundles, together with an example which we will see later, were given by Raynaud in [33]. A useful survey of the area by Beauville can be found in [4]. More examples of bundles without theta divisors were found by Popa in [28], and Schneider in [36] gives results on the dimension of the locus of such bundles in some of the moduli spaces $\mathcal{U}(r, r(g-1))$.

In this paper we focus on the case of semistable symplectic bundles of rank four (so $n = 2$) over a curve of genus two. Here \mathcal{M}_2 is of dimension ten and D is a rational map $\mathcal{M}_2 \dashrightarrow |4\Theta|_+ = |\Xi|^* = \mathbb{P}^9$. Some time ago, Arnaud Beauville found that some of Raynaud's bundles in this case admit symplectic structures, and conjectured that these were the only possibilities in this situation. The aim of this paper is to prove Beauville's conjecture:

Theorem 3 *If X is of genus two then the base locus of the linear system $|\Xi|$ on \mathcal{M}_2 consists of six points, which are in canonical bijection with the Weierstrass points of X .*

Here is a summary of the paper. We begin with two results for genus g . One describes certain isotropic subsheaves of stable symplectic bundles of rank $2n$ without theta divisors. The second is that in rank four, any base points of $|\Xi|$ must be stable vector bundles.

We then specialise to $g = 2$. Firstly, we show that the expected number of base points is six in this case.

An important ingredient is a description of this \mathcal{M}_2 from [13] using vector bundle extensions. Let $W \rightarrow X$ be a symplectic or orthogonal bundle of

rank $2n$ and $E \subset W$ an *Lagrangian* subbundle (that is, isotropic of maximal rank n). Then W is naturally an extension of E^* by E , and defines a class $\delta(W) \in H^1(X, \text{Hom}(E^*, E))$. Conversely, it is natural to ask which such extensions are induced by bilinear forms. We have

Theorem 4 *Let W be an extension of E^* by E . Then W carries a symplectic (resp., orthogonal) structure with respect to which E is isotropic if and only if W is isomorphic as a vector bundle to an extension whose class belongs to the subspace $H^1(X, \text{Sym}^2 E)$ (resp., $H^1(X, \bigwedge^2 E)$ of $H^1(X, \text{Hom}(E^*, E))$).*

Proof

See [12], Criterion 2.

This motivates the following result:

Theorem 5 *Let X be a curve of genus two. Every semistable symplectic vector bundle of rank four over X is an extension $0 \rightarrow E \rightarrow W \rightarrow E^* \rightarrow 0$ for some stable bundle $E \rightarrow X$ of rank two and degree -1 , where E is isotropic and the class $\delta(W)$ is symmetric. Moreover, for generic W , there are 24 such E . Equivalently, the moduli map*

$$\Phi: \bigcup_{\substack{E \text{ stable of rank two} \\ \text{and degree } -1}} \mathbb{P}H^1(X, \text{Sym}^2 E) \dashrightarrow \mathcal{M}_2$$

is a surjective morphism¹ which is generically finite of degree 24.

Proof

This is the main result of [13].

We denote the five-dimensional projective space $\mathbb{P}H^1(X, \text{Sym}^2 E)$ by \mathbb{P}_E^5 . By Theorem 5, it suffices to search for base points of $|\Xi|$ in each \mathbb{P}_E^5 separately. In § 5, we construct a rational map $J^1 \dashrightarrow (\mathbb{P}_E^5)^*$ with some useful properties, and use it to prove that no \mathbb{P}_E^5 contains more than one base point of $|\Xi|$.

In § 6, we consider extensions $0 \rightarrow O_X(-w) \rightarrow E \rightarrow O_X \rightarrow 0$ where w is a Weierstrass point of X . We show that this \mathbb{P}_E^5 contains a bundle W_w without a theta divisor, and furthermore (§ 7), that the isomorphism class of W_w does not depend on the class of the extension E in $H^1(X, \text{Hom}(O_X, O_X(-w)))$.

In § 8, we prove that any $W \in \mathcal{M}_2$ without a theta divisor must contain some such family of extensions E as isotropic subbundles, so is of this form.

¹We must of course give this union of projective spaces a suitable algebraic structure.

To conclude, we show that the base locus is reduced, so consists of six points.

In § 9, we recall a construction of Raynaud [33] of stable bundles of rank four and degree zero over X which have no theta divisors, and describe Beauville's construction of symplectic structures on some of these bundles. We then show how the bundles constructed in § 6 correspond to these ones.

In the last section, we use Thm. 3 to show that $D: \mathcal{M}_2 \dashrightarrow |4\Theta|_+$ is surjective, and notice that there exist stable bundles in \mathcal{M}_2 with reducible theta divisors.

2 Some isotropic subsheaves

In this section, X will have genus $g \geq 2$. Firstly, we quote a couple of technical results. Let F and G be vector bundles over X . We describe two maps between associated cohomology spaces. The first one is the cup product

$$\cup: H^1(X, \operatorname{Hom}(G, F)) \rightarrow \operatorname{Hom}(H^0(X, G) \rightarrow H^1(X, F)),$$

and the second the natural multiplication map of sections

$$m: H^0(X, G) \otimes H^0(X, K_X \otimes F^*) \rightarrow H^0(X, K_X \otimes F^* \otimes G).$$

Proposition 6 *The maps \cup and m are canonically dual, via Serre duality.*

Proof

This is well known; see for example [11], chap. 6.

We will also use the map $c: H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \operatorname{End}(M \otimes W))$ induced by $\lambda \mapsto \lambda \cdot \operatorname{Id}_{M \otimes W}$.

Notation: We denote the sheaf of regular sections of a vector bundle W , L , \mathcal{O}_X , K_X by the corresponding script letter \mathcal{W} , \mathcal{L} , \mathcal{O}_X , \mathcal{K}_X .

We now give a result for symplectic bundles of rank $2n$. We will need an adaptation of Prop. 2.6 (1) in Mukai [24]. Recall that the tangent space to J^{g-1} at any point is isomorphic to $H^1(X, \mathcal{O}_X)$. The number $h^0(X, L \otimes W)$ is constant on an open subset of $S(W)$. Let M belong to this subset.

Proposition 7 *The tangent space to $S(W)$ at M is isomorphic to*

$$\operatorname{Ker} (\cup \circ c: H^1(X, \mathcal{O}_X) \rightarrow \operatorname{Hom}(H^0(X, M \otimes W), H^1(X, M \otimes W))).$$

Proof

Recall that a deformation of $M \otimes W$ is an exact sequence

$$0 \rightarrow M \otimes W \rightarrow \mathbf{V} \rightarrow M \otimes W \rightarrow 0.$$

We are interested in those deformations which are of the form $c(v)$. By definition, $c(v)$ is tangent to the locus $S(W)$ if and only if all global sections of $M \otimes W$ lift to the extension $\mathbf{V}_{c(v)}$. But we have the cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(X, M \otimes W) \rightarrow H^0(X, \mathbf{V}_{c(v)}) \rightarrow H^0(X, M \otimes W) \\ \rightarrow H^1(X, M \otimes W) \rightarrow \dots \end{aligned}$$

Following Kempf [15], one shows that the boundary map is none other than cup product by $c(v)$. Therefore, *all* global sections of $M \otimes W$ lift if and only if cup product by $c(v)$ is zero, that is, $v \in \text{Ker}(\cup \circ c)$. \square

Now we can prove

Theorem 8 *Let W be a stable symplectic bundle W of rank $2n$ over X . Then $h^0(X, L \otimes W) > 0$ for every $L \in J^{g-1}$ if and only if \mathcal{W} has an isotropic subsheaf $\mathcal{L}^{-1} \oplus \mathcal{K}_X^{-1}\mathcal{L}$ for generic $L \in J^{g-1}$.*

Proof

The “if” is clear. Conversely, suppose $h^0(X, L \otimes W) > 0$ for all $L \in J^{g-1}$. Let $M \in S(W)$ be such that $h^0(X, M \otimes W)$ is the generic value. Then the tangent space to $S(W)$ at M is the whole of $H^1(X, \mathcal{O}_X)$. By Prop. 7, the map $\cup \circ c$ is zero.

Now by Prop. 6, the map \cup is dual to the multiplication

$$H^0(X, M \otimes W) \otimes H^0(X, K_X M^{-1} \otimes W^*) \rightarrow H^0(X, \text{Hom}(K_X \otimes W^* \otimes W)).$$

Moreover, one can show that c is dual to the trace map

$$\text{tr}: H^0(X, \text{Hom}(M \otimes W, K_X \otimes (M \otimes W))) \rightarrow H^0(X, K_X).$$

Since $\cup \circ c$ is zero, then, so is $\text{tr} \circ m$.

Now the trace map can be identified with the map defined on decomposable elements by $e^* \otimes f \mapsto e^*(f)$. Also, W is self-dual via the symplectic form ω , and the induced isomorphism $W \xrightarrow{\sim} W^*$ is unique up to scalar since W is stable.

Combining these facts, the vanishing of $\text{tr} \circ m$ means that the images of the maps $M^{-1} \rightarrow W$ and $K_X^{-1}M \rightarrow W$ annihilate under contraction. Moreover, any line bundle is isotropic with respect to a symplectic form. If $M^2 \neq K_X$, then, there is an isotropic subsheaf $\mathcal{M}^{-1} \oplus \mathcal{K}_X^{-1}\mathcal{M}$ in \mathcal{W} . This completes the proof of Thm. 8. \square

3 Stability of symplectic bundles without theta divisors

Henceforth, we assume that $n = 2$. We will need

Proposition 9 *Let X be a curve of genus $g \geq 2$ and $F \rightarrow X$ a semistable bundle of rank at most two and degree zero. Then $h^0(X, L \otimes F) = 0$ for generic $L \in J^{g-1}$.*

Proof

This follows from Raynaud [33], Prop. 1.6.2.

Lemma 10 *Let X be a curve of genus $g \geq 2$. Any semistable symplectic bundles of rank four over X without theta divisors are in fact stable vector bundles.*

Proof

We show that every strictly semistable symplectic vector bundle W over X of rank four admits a theta divisor. It can be shown² that such a W is S -equivalent to a direct sum of stable bundles of rank one and/or two and degree zero. Thus it suffices to prove that every such direct sum admits a theta divisor. This follows from the last proposition. \square

4 The number of base points in the genus two case

For the rest of the paper, we suppose that X has genus two. In this section we find the expected number of base points of $|\Xi|$ in this case.

Determinant bundles

Here we recall very briefly some facts about line bundles over the moduli space $\mathcal{M}(\mathrm{Sp}_2\mathbb{C})$. For a general treatment of this kind of question, we refer to Beauville–Laszlo–Sorger [7], Laszlo–Sorger [19] and Sorger [37].

To a representation of $\mathrm{Sp}_2\mathbb{C}$, we can associate a line bundle over $\mathcal{M}(\mathrm{Sp}_2\mathbb{C})$, called the *determinant bundle* of the representation. Ξ is the determinant bundle of the standard representation of $\mathrm{Sp}_2\mathbb{C}$, and the Picard group of

²This is proven in [11], chap. 2; the arguments are adapted from those of Ramanan [29] and Ramanathan [32] for orthogonal bundles.

$\mathcal{M}(\mathrm{Sp}_2\mathbb{C})$ is $\mathbb{Z} \cdot \Xi$. To a representation ρ of $\mathrm{Sp}_2\mathbb{C}$ we associate a number d_ρ called the *Dynkin index* of ρ , and the determinant bundle of ρ is Ξ^{d_ρ} . The canonical bundle of $\mathcal{M}(\mathrm{Sp}_2\mathbb{C})$ is the dual of the determinant bundle of the adjoint representation, and is therefore Ξ^{-6} by Sorger [37], Tableau B, since $\mathrm{Sp}_2\mathbb{C}$ is of type C_2 .

Proposition 11 *If the base locus of $|\Xi|$ is of dimension zero then it consists of six points, counted with multiplicity.³*

Proof

The dimension of \mathcal{M}_2 is ten, so if the base locus of $|\Xi|$ is of dimension zero then its scheme theoretic length is given by $c_1(\Xi)^{10}$. To calculate this number, we follow an approach of Laszlo [18], § V, Lemma 5. The Hilbert function of Ξ is defined as $n \mapsto \chi(\mathcal{M}_2, \Xi^n)$. For large enough n , this coincides with a polynomial $p(n)$. We claim that the leading term of $p(n)$ is $c_1(\Xi)^{10}/10!$. To see this, suppose α is the Chern root of Ξ . Then, by Hirzebruch–Riemann–Roch,

$$\chi(\mathcal{M}_2, \Xi^n) = \int_{\mathcal{M}_2} \exp(n\alpha) \mathrm{td}(\mathcal{M}_2).$$

Since $\alpha^i = 0$ for all $i \geq 11$, the only term which contains n^{10} here is $c_1(\Xi)^{10}/10!$ as required.

Now we have seen that the canonical bundle of \mathcal{M}_2 is Ξ^{-6} . Hence, by Serre duality, $p(n) = (-1)^{10} \chi(\mathcal{M}_2, \Xi^{-6-n}) = p(-6-n)$, equivalently, $p(n)$ is symmetric about $n = -3$.

By a result stated on p. 4 of Oxbury [26], the spaces $H^i(\mathcal{M}_2, \Xi^n)$ vanish for all $i > 0$ and $n > 0$. Moreover, for all $n < 0$, we have $h^0(\mathcal{M}_2, \Xi^n) = 0$ by stability. Thus $p(-5) = p(-4) = p(-3) = p(-2) = p(-1) = 0$. Hence $p(n)$ is equal to

$$\gamma(n+5)(n+4)(n+3)^2(n+2)(n+1)(n-\alpha)(n+6+\alpha)(n-\beta)(n+6+\beta)$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. We wish to find γ . To do this, we find the values of p at 0, 1 and 2. By the Verlinde formula (Oxbury–Wilson [27], § 2) we have $p(0) = 1$ and $p(1) = 10$ and

$$p(2) = 2^2 \times 5^2 \times \sum \mathcal{S}(s, t)^{-2}$$

where

$$\mathcal{S}(s, t) = 2^4 \sin\left(\frac{\pi(s+t)}{5}\right) \sin\left(\frac{\pi t}{5}\right) \sin\left(\frac{\pi s}{10}\right) \sin\left(\frac{\pi(t+2s)}{10}\right)$$

³This calculation was first done by Arnaud Beauville.

and the sum is taken over all pairs s, t with $s, t \geq 1$ and $s + t \leq 4$. We calculate $p(2) = 58$ with Maple. These values yield the equations

$$\begin{aligned}\gamma \cdot 5! \times 3(-\alpha)(6 + \alpha)(-\beta)(6 + \beta) &= 1, \\ \gamma \cdot 6! \times 3(1 - \alpha)(7 + \alpha)(1 - \beta)(7 + \beta) &= 10, \\ \gamma \cdot \frac{7!}{2} \times 3(2 - \alpha)(8 + \alpha)(2 - \beta)(8 + \beta) &= 58.\end{aligned}$$

Solving with Maple, we obtain $\gamma = 6 \times 10!^{-1}$, so if $\text{Bs}|\Xi|$ is of dimension zero then it consists of six points, counted with multiplicity. \square

5 Study of the extension spaces

We now begin to study the extension spaces $\mathbb{P}_E^5 = \mathbb{P}H^1(X, \text{Sym}^2 E)$ where E is a stable bundle of rank two and degree -1 . We describe a rational map $J^1 \dashrightarrow (\mathbb{P}_E^5)^*$. Firstly, we state a result similar to Thm. 5:

Lemma 12 *Every stable bundle $E \rightarrow X$ of rank two and degree -1 is a nontrivial extension $0 \rightarrow L^{-1} \rightarrow E \rightarrow M^{-1} \rightarrow 0$ for one, two, three or (generically) four pairs (L, M) where L and M are line bundles of degrees one and zero respectively. Moreover, $h^0(X, \text{Hom}(L^{-1}, E)) = 1 = h^0(X, \text{Hom}(E, M^{-1}))$ for all such (L, M) .*

Proof

This follows from [13], Lemmas 5 and 6.

We claim that $h^0(X, L \otimes E^*) \cdot h^0(X, K_X L^{-1} \otimes E^*) = 1$ for generic $L \in J^1$. To see this, note that by Riemann–Roch, $h^0(X, L \otimes E^*) \geq 2$ if and only if $h^1(X, L \otimes E^*)$ is nonzero. By Serre duality,

$$h^1(X, L \otimes E^*) = h^0(X, K_X L^{-1} \otimes E).$$

But by Lemma 12, this is nonzero for at most four L . In a similar way, we see that $h^0(X, K_X L^{-1} \otimes E^*)$ is greater than 1 for at most four L . Thus

$$h^0(X, L \otimes E^*) \cdot h^0(X, K_X L^{-1} \otimes E^*)$$

is different from 1 for at most eight L . We write U_E for the complement of these points in J^1 .

For each $L \in U_E$, we can consider the composed map \tilde{m}

$$\begin{array}{ccc} H^0(X, L \otimes E^*) \otimes H^0(X, K_X L^{-1} \otimes E^*) & \xrightarrow{m} & H^0(X, K_X \otimes E^* \otimes E^*) \\ & & \downarrow \\ & & H^0(X, K_X \otimes \text{Sym}^2 E^*). \end{array}$$

We claim that the image of \tilde{m} is of dimension 1; this follows from the last paragraph and the fact that no nonzero decomposable vector is antisymmetric. Thus we can define a rational map

$$j_E: J^1 \dashrightarrow \mathbb{P}H^0(X, K_X \otimes \text{Sym}^2 E^*) = (\mathbb{P}_E^5)^*$$

by sending L to the image of \tilde{m} . This is defined exactly on U_E .

Now by Serre duality, a nontrivial symplectic extension W of E^* by E with class $\delta(W) \in H^1(X, \text{Sym}^2 E)$ defines a hyperplane $H_W = \mathbb{P} \text{Ker}(\delta(W))$ in $(\mathbb{P}_E^5)^*$. We have

Lemma 13 *Let $W \in \mathbb{P}_E^5$ be a symplectic extension and $L \rightarrow X$ a line bundle of degree one belonging to $U_E \subset J^1$. Then $h^0(X, L \otimes W) > 0$ if and only if $j_E(L) \in H_W$.*

Proof

Tensoring the sequence $0 \rightarrow E \rightarrow W \rightarrow E^* \rightarrow 0$ by L , we get the cohomology sequence

$$0 \rightarrow H^0(X, L \otimes W) \rightarrow H^0(X, L \otimes E^*) \xrightarrow{\cdot \cup \delta(W)} H^1(X, L \otimes E) \rightarrow \dots$$

whence $h^0(X, L \otimes W) > 0$ if and only if the cup product map has a kernel. By hypothesis,

$$h^0(X, K_X L^{-1} \otimes E) = h^1(X, L \otimes E^*) = 0$$

so $h^0(X, L \otimes E^*) = 1$ by Riemann–Roch. Similarly, we see that

$$h^1(X, L \otimes E) = h^0(X, K_X L^{-1} \otimes E^*) = 1.$$

Thus $h^0(X, L \otimes W) > 0$ if and only if cup product by $\delta(W)$ is zero. By Prop. 6 (with $F = L \otimes E$ and $G = L \otimes E^*$), this means that $m^* \delta(W) = 0$. In other words, the image of

$$m: H^0(X, L \otimes E^*) \otimes H^0(X, K_X L^{-1} \otimes E^*) \rightarrow H^0(X, K_X \otimes E^* \otimes E^*)$$

belongs to $\text{Ker}(\delta(W))$. This is equivalent to $\text{Im}(\tilde{m}) \subset \text{Ker}(\delta(W))$ because $\delta(W)$ is symmetric. Projectivising, this becomes $j_E(L) \in H_W$ (our hypothesis of generality on L implies that $j_E(L)$ is defined). \square

By Lemma 13, we see that \mathbb{P}_E^5 contains an extension without a theta divisor if and only if the image of j_E is contained in a hyperplane $H \subset (\mathbb{P}_E^5)^*$. The extension will be that W such that $H_W = H$.

Lemma 14 *For a general stable $E \rightarrow X$ of rank two and degree -1 , the image of j_E is nondegenerate in $(\mathbb{P}_E^5)^*$, and for any such E , it spans a hyperplane.*

Proof

By Lemma 12, we have at least one short exact sequence

$$0 \rightarrow M \rightarrow E^* \xrightarrow{b} N \rightarrow 0 \quad (1)$$

where M and N are line bundles of degree zero and one respectively. We claim that this induces a short exact sequence

$$0 \rightarrow K_X M^2 \rightarrow K_X \otimes \text{Sym}^2 E^* \xrightarrow{c} K_X N \otimes E^* \rightarrow 0 \quad (2)$$

where c is induced by the map $E \otimes E \rightarrow N \otimes E$ given by

$$e \otimes f \mapsto b(e) \otimes f + b(f) \otimes e.$$

We work with the map induced by c on the associated locally free \mathcal{O}_X -modules. Let $e, f \in \mathcal{E}_x$ be such that $e \in \mathcal{M}_x$ but f is not. Then the image of $e \otimes f + f \otimes e$ belongs to $\mathcal{N} \otimes M$ but that of $f \otimes f$ does not. Thus the image contains two linearly independent elements of $\mathcal{N} \otimes E$, so the map on sheaves is surjective. The kernel of c clearly contains M^2 , so is equal to it since they are of the same rank and degree. This establishes the claim.

Now note that the class $\langle \delta(E) \rangle \in \mathbb{P}H^1(X, \text{Hom}(L, M))$ can be identified with a divisor $p_1 + p_2 + p_3 \in |K_X LM^{-1}|$. We make the following hypotheses of generality on E :

- At least one degree zero line subbundle $M \subset E^*$ is not a point of order two in J^0 .
- For each such M , there is a unique pair of points $q_1, q_2 \in X$ such that

$$K_X M^2 = \mathcal{O}_X(q_1 + q_2).$$

We require that for at least one such $M \subset E$, the sets $\{q_1, q_2\}$ and $\{\iota p_1, \iota p_2, \iota p_3\}$ be disjoint.

We consider a short exact sequence (1) where M^2 is nontrivial. The associated cohomology sequence is then

$$0 \rightarrow H^0(X, K_X M^2) \rightarrow H^0(X, K_X \otimes \text{Sym}^2 E^*) \rightarrow H^0(X, K_X N \otimes E^*) \rightarrow 0.$$

We show that $\mathbb{P}H^0(X, K_X M^2)$ is spanned by points of $j_E(U_E)$. It is not hard to see that $j_E(L)$ is this point if and only if $L \in U_E$ and

$$L^{-1} = M(-x) \quad \text{and} \quad K_X^{-1}L = M(-y)$$

for some points x, y of the curve. This condition can be interpreted geometrically as $L \in t_{M^{-1}}\Theta \cap t_M\Theta$ (notice that in genus two there is a canonical isomorphism $X \xrightarrow{\sim} \text{Supp}(\Theta)$). Since M^2 is nontrivial, this consists generically of $\Theta^2 = 2$ points, which are exchanged by ι . We take

$$L = M(\iota q_1) = M^{-1}(q_2)$$

where q_1 and q_2 are as defined above; then $K_X L^{-1} = M(\iota q_2) = M^{-1}(q_1)$. We check that j_E is defined at these points. It is necessary that neither $M(\iota q_1) = M^{-1}(q_2)$ nor its image $M(\iota q_2) = M^{-1}(q_1)$ under the Serre involution be a quotient of E^* or the Serre image of a quotient of E^* . It is not hard to show⁴ that the degree one line bundle quotients of E^* are L^{-1} , $M^{-1}(p_1)$, $M^{-1}(p_2)$ and $M^{-1}(p_3)$. Thus we must check that

$$\{M(\iota q_1), M(\iota q_2)\} \cap \{N, K_X N^{-1}, M(p_i), K_X M^{-1}(-p_i)\} = \emptyset.$$

Since the set where j_E is not defined is ι -invariant, it suffices to check that neither $M(\iota q_1)$ nor $M(\iota q_2)$ is equal to N or $M(p_i)$.

If $M(\iota q_j) = N$ then $\mathbb{P}H^1(X, \text{Hom}(N, M)) \cong |K_X(\iota q_j)|$ and

$$p_1 + p_2 + p_3 = \iota q_j + r + \iota r$$

for some $r \in X$. Thus $q_j = \iota p_i$ for some i, j . On the other hand, if $M(\iota q_j) = M(p_i)$ then $q_j = \iota p_i$. By our second assumption of generality, then, j_E is defined at both $M(\iota q_j)$, and $\mathbb{P}H^0(X, K_X M^2)$ belongs to $j_E(U_E)$.

Now K_X tensored with (1) yields the cohomology sequence

$$0 \rightarrow H^0(X, K_X M) \rightarrow H^0(X, K_X \otimes E^*) \rightarrow H^0(X, K_X N) \rightarrow 0$$

⁴See for example the proof of [13], Lemma 6.

and we have the diagram

$$\begin{array}{ccccc}
& & 0 & & \\
& & \downarrow & & \\
& & H^0(X, K_X NM) & & \\
& & \downarrow & & \\
H^0(X, K_X \otimes \text{Sym}^2 E^*) & \xrightarrow{c} & H^0(X, K_X N \otimes E^*) & \longrightarrow & 0 \\
& \searrow d & \downarrow & & \\
& & H^0(X, K_X N^2) & & \\
& & \downarrow & & \\
& & 0 & &
\end{array}$$

We must show that the image of $\mathbb{P}c \circ j_E$ is nondegenerate in $\mathbb{P}H^0(X, K_X N \otimes E^*)$. For this, it suffices to show that $\mathbb{P}H^0(X, K_X NM)$ is spanned and that $\mathbb{P}d \circ j_E(U_E)$ is nondegenerate in $\mathbb{P}H^0(X, K_X N^2)$.

By the definition of c , we have $\mathbb{P}c \circ j_E(L) \in \mathbb{P}H^0(X, K_X NM)$ if $L^{-1} = M(-x)$ for some $x \in X$ but $K_X^{-1}L \neq M(-y)$ for any $y \in X$, or vice versa. This is equivalent to L belonging to the symmetric difference of $t_{M^{-1}}\Theta$ and $t_M\Theta$. Since $M \neq M^{-1}$, we can find infinitely many such L . We need to find two which define different divisors in $|K_X NM|$. We observe that if x is not a base point of $|K_X NM|$ then $\mathbb{P}c \circ j_E(M^{-1}(x))$ is the divisor in $|K_X NM|$ containing x . Thus if neither x nor y is a base point and they belong to different divisors of $|K_X NM|$ then the images of $j_E(M^{-1}(x))$ and $j_E(M^{-1}(y))$ generate $\mathbb{P}H^0(X, K_X NM)$.

Next, we have $\mathbb{P}d \circ j_E(L) \in \mathbb{P}H^0(X, K_X N^2)$ if and only if neither L^{-1} nor $K_X^{-1}L$ is of the form $M(-x)$, equivalently, L does not belong to $t_{M^{-1}}\Theta \cup t_M\Theta$. In fact $\mathbb{P}d \circ j_E$ is dominant. Let $x_1 + x_2 + x_3 + x_4$ be any divisor in $|K_X N^2|$. For generic x_1 and x_2 , we know that $N^{-1}(x_1 + x_2)$ will not belong to $t_{M^{-1}}\Theta \cup t_M\Theta$, so we can put $L = N^{-1}(x_1 + x_2)$ and then

$$K_X L^{-1} = K_X N(-x_1 - x_2) = N^{-1}(x_3 + x_4).$$

Thus $\mathbb{P}d \circ j_E$ is dominant (and generically of degree six). In particular, the image spans $|K_X N^2| = \mathbb{P}^2$.

Finally, we show that the image of j_E always spans a \mathbb{P}^4 . If E^* fits into a short exact sequence $0 \rightarrow M \rightarrow E^* \rightarrow N \rightarrow 0$ where M is of order two, then

we form as before a cohomology diagram

$$\begin{array}{ccccc}
& & 0 & & \\
& & \downarrow & & \\
& & H^0(X, K_X NM) & & \\
& & \downarrow & & \\
H^0(X, K_X \otimes \text{Sym}^2 E) & \xrightarrow{c} & H^0(X, K_X N \otimes E) & \longrightarrow & \dots \\
& \searrow d & \downarrow & & \\
& & H^0(X, K_X N^2) & & \\
& & \downarrow & & \\
& & 0 & &
\end{array}$$

Since $M = M^{-1}$, we have $t_{M^{-1}}\Theta = t_M\Theta$ and j_E is defined at infinitely many points of $t_{M^{-1}}\Theta \cap t_M\Theta$. Choose two distinct points $p, q \in X$ such that $p+q$ is not a canonical divisor; then $j_E(M(p))$ and $j_E(M(q))$ span $\mathbb{P}H^0(X, K_X M^2)$. Since the symmetric difference of $t_{M^{-1}}\Theta$ and $t_M\Theta$ is empty, there are no points of the image mapping to $\mathbb{P}H^0(X, K_X NM) \subset \mathbb{P}H^0(X, K_X N \otimes E)$. However, the composed map to $\mathbb{P}H^0(X, K_X N^2)$ is dominant, by the same argument. Thus we can find five linearly independent points of the image of j_E , spanning a \mathbb{P}^4 in $(\mathbb{P}_E^5)^*$.

On the other hand, if the second hypothesis of generality fails for some exact sequence $0 \rightarrow M \rightarrow E^* \rightarrow N \rightarrow 0$ where M is not of order two, then j_E is not defined at either $M(\iota q_j)$. However, the rest of the proof goes through and again we can find five independent points of the image of $j_E(U_E)$. \square

Remark: In fact the second generality hypothesis can be weakened slightly (we then have to blow up J^1 at a point) but the statement as given is strong enough for our purposes.

6 An example of a base point

In this section we give an explicit example of a base point of $|\Xi|$. We begin by constructing an E which violates both of the generality conditions stated in the last lemma.

Let $w \in X$ be a Weierstrass point. Extensions

$$0 \rightarrow O_X(-w) \rightarrow E \rightarrow O_X \rightarrow 0 \quad (3)$$

are determined as vector bundles by a divisor in

$$|K_X(w)| \cong \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-w))).$$

This divisor will be of the form $w+p+\iota p$ where ι is the hyperelliptic involution on X . It is not hard to see that the degree -1 subbundles of E are $O_X(-w)$, $O_X(-p)$ and $O_X(-\iota p)$.

Proposition 15 *The bundle E is ι -invariant.*

Proof

By pulling back the sequence (3) by ι , we get a short exact sequence

$$0 \rightarrow O_X(-w) \rightarrow \iota^*E \rightarrow O_X \rightarrow 0$$

since $O_X(-w)$ and O_X are ι -invariant. The divisor determining the extension ι^*E is $\iota(w) + \iota(p) + \iota(\iota p) = w + \iota p + p$, so ι^*E is isomorphic to E . \square

We now give some more information on D . For any stable $E \rightarrow X$ of rank two and degree -1 , we have a map $D_E: \mathbb{P}_E^5 \dashrightarrow |4\Theta|_+$ which is the composition of $D: \mathcal{M}_2 \dashrightarrow |4\Theta|_+$ with the classifying map $\Phi|_{\mathbb{P}_E^5}: \mathbb{P}_E^5 \rightarrow \mathcal{M}_2$.

Lemma 16 *The map $D_E: \mathbb{P}_E^5 \dashrightarrow |4\Theta|_+$ is linear.*

Proof

The degree of D_E is constant with respect to E since the moduli space of such E is connected (see for example Le Potier [21]). It therefore suffices to prove the lemma for a general E . In particular, we can suppose that D_E is defined everywhere on \mathbb{P}_E^5 .

For $L \in J^1$, let $H_L \subset |4\Theta|_+$ denote the hyperplane of divisors containing L . By definition, $D_E\langle\delta(W)\rangle$ belongs to H_L if and only if $h^0(X, L \otimes W) > 0$. We show that this is a linear condition on \mathbb{P}_E^5 . Let W be an extension of E^* by E . We get a cohomology sequence

$$0 \rightarrow H^0(X, L \otimes W) \rightarrow H^0(X, L \otimes E^*) \xrightarrow{\cup_L \delta(W)} H^1(X, L \otimes E) \rightarrow H^1(X, L \otimes W) \rightarrow 0.$$

If $h^0(X, L \otimes E^*) \geq 2$ then $h^0(X, K_X L^{-1} \otimes E) \geq 1$ and the image of D_E is contained in $H_{K_X L^{-1}} = H_L$, so $D_E^* H_L$ is the whole space.

If $h^0(X, L \otimes E^*) = 1$ then $H^0(X, L \otimes W)$ is nonzero if and only if $\cup_L \delta(W)$ is the zero map. Thus, set-theoretically, $D_E^* H_L = \cup_L^{-1}(0)$. Now \cup_L is a linear map

$$H^1(X, \text{Sym}^2 E^*) \rightarrow \text{Hom}(H^0(X, L \otimes E), H^1(X, L \otimes E)).$$

Since the latter space has dimension 1, $\text{Ker}(\cup_L)$ is a hyperplane.

It remains to show that the multiplicity of $D_E^*H_L$ is 1. We begin by showing that $D_E(\mathbb{P}_E^5)$ is contained in a \mathbb{P}^5 in $|4\Theta|_+$. By generality of E , we can suppose that E has four distinct line subbundles of degree -1 not including any pairs of the form $N, K_X^{-1}N^{-1}$. Thus the divisors in the image of D_E all contain four points distinct modulo ι . We show that these impose independent conditions on $|4\Theta|_+$.

Let $L^{-1} \subset E$ be a line subbundle of degree -1 and M^{-1} the quotient of E by L^{-1} . Then E is defined by a divisor $p_1 + p_2 + p_3 \in |K_X ML|$. By generality, we can suppose that this linear system is base point free and that the p_i are distinct. The other degree -1 subbundles of E are then $M^{-1}(-p_1)$, $M^{-1}(-p_2)$ and $M^{-1}(-p_3)$. We construct even 4Θ divisors D_0, D_1, \dots, D_4 containing none, one, two, three and all of these points respectively. Since $\phi_{4\Theta_+}: J^1 \rightarrow |4\Theta|_+^*$ descends to an embedding of the Kummer variety and the points $L, M(p_i)$ are distinct on the Kummer, we can easily find D_0 and D_1 . For D_2 , we use the fact that $\phi_{2\Theta}$ also gives an embedding of the Kummer. Thus we can find a 2Θ divisor G containing $M(p_1)$ but none of other points, and another, G' , containing $M(p_2)$ but none of the others. Since every 2Θ divisor is even, the sum $G + G'$ is an even 4Θ divisor containing exactly two of the points. For D_3 , we take $2(t_M\Theta + t_{M^{-1}}\Theta)$. By generality, L is not of the form either $M(x)$ or $M^{-1}(x)$ for any $x \in X$, so this divisor contains the three $M(p_i)$ but not L . Finally, choose any symplectic extension W of E^* by E which has a theta divisor $D(W)$; these exist by Lemma 14. Then we take $D_4 = D(W)$; this contains all the points.

The four degree -1 line subbundles of E thus impose independent conditions on the divisors in $|4\Theta|_+ = \mathbb{P}^9$. Hence the image of D_E is contained in a \mathbb{P}^5 . By generality and Lemma 14, the map D_E is a morphism $\mathbb{P}^5 \rightarrow \mathbb{P}^5$, so must be surjective, therefore a finite cover. If not an isomorphism, it must be branched over a hypersurface. Now the image of $\phi_{4\Theta_+}: J^1 \rightarrow |4\Theta|_+^*$ is nondegenerate, so we can find an H_L whose intersection with the image of D_E is not contained in this branch locus. Then $D_E^*H_L$ is reduced, and we have seen that its support is a hyperplane.

This completes the proof of Lemma 16. \square

Now since E is ι -invariant, we can lift ι to linearisations $\tilde{\iota}$ and $\text{Sym}^2(\tilde{\iota})$ of E and $\text{Sym}^2 E$. The latter acts on $H^1(X, \text{Sym}^2 E)$, taking the class of an extension W to that of ι^*W (modulo a scalar). Since this is an involution, we can decompose $H^1(X, \text{Sym}^2 E)$ into $+1$ and -1 eigenspaces $H^1(X, \text{Sym}^2 E)_\pm$.

Suppose now that E is of the form (3). Let

$$\widetilde{D}_E: H^1(X, \text{Sym}^2 E) \rightarrow H^0(J^1, 4\Theta)_+$$

be a linear lift of D_E . The kernel of \widetilde{D}_E consists of exactly the extensions without theta divisors. Now when $D(W)$ exists, we have $\iota^* D(W) = D(\iota^* W)$ since their supports are equal and they are both even 4Θ divisors. Therefore, for all $W \in \mathbb{P}_E^5$, either

$$\widetilde{D}_E(\delta(\iota^* W)) = \widetilde{D}_E(\delta(W)) \quad \text{or} \quad \widetilde{D}_E(\delta(\iota^* W)) = -\widetilde{D}_E(\delta(W)).$$

This means that one of the spaces $H^1(X, \text{Sym}^2 E)_\pm$ belongs to the kernel of \widetilde{D}_E . We calculate the dimensions of these eigenspaces. The key tool is the fixed point formula of Atiyah and Bott:

Theorem 17 *Let M be a compact complex manifold and $\gamma: M \rightarrow M$ an automorphism with a finite set $\text{Fix}(\gamma)$ of fixed points. Suppose that γ lifts to a linearisation $\tilde{\gamma}$ of a holomorphic vector bundle $V \rightarrow M$. Then*

$$\sum_j (-1)^j \text{tr}(\gamma|_{H^j(M, V)}) = \sum_{p \in \text{Fix}(\gamma)} \frac{\text{tr}(\gamma|_{E|_p})}{\det(I - d\gamma_p)}.$$

Proof

See Atiyah–Bott [2], Theorem 4.12.

For us $M = X$, with $\gamma = \iota$, and $V = \text{Sym}^2 E$. We write down the linearisation $\tilde{\iota}$ of E explicitly and determine the action of $\text{Sym}^2(\tilde{\iota})$ on the fibre of $\text{Sym}^2 E$ over the Weierstrass points. For any bundle V with sheaf of sections \mathcal{V} , the fibre of V at a point x is identified with $\mathcal{V}_x/m_x \mathcal{V}_x$ where m_x is the maximal ideal of the ring $\mathcal{O}_{X,x}$ (see for example Le Potier [21], chap. 1). Let z be a uniformiser at w . Then ι is given near w by $z \mapsto -z$.

Now it is not hard to show that a linearisation of an involution on a line bundle is unique up to multiplication by -1 . Modulo this, the last paragraph shows:

- A lift of ι to \mathcal{O}_X acts trivially on $\mathcal{O}_X|_p$ for all Weierstrass points p .
- Since $\mathcal{O}_X(-w)|_p \cong \mathcal{O}_X|_p$ for each $p \neq w$, the same is true for a lift of ι to $\mathcal{O}_X(-w)$ at $\mathcal{O}_X(-w)|_p$ for these p .
- A lift of ι to $\mathcal{O}_X(-w)$ acts by -1 on $\mathcal{O}_X(-w)|_w$.

Near the point w , the bundle E looks like $O_X(-w) \oplus O_X$. This is far from canonical, but we are interested only in the trace of a linearisation, which is independent of the trivialisation. We normalise $\tilde{\iota}$ such that the induced linearisation on $O_X(-w)$ acts on the fibre over a Weierstrass point p by

$$\begin{cases} -1 & \text{if } p \neq w \\ +1 & \text{if } p = w. \end{cases}$$

Then $\tilde{\iota}$ acts on the fibres $E|_p$ by either

$$\begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } p \neq w \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } p = w \end{cases} \quad \text{or} \quad \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } p \neq w \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } p = w. \end{cases}$$

Using the sequence $0 \rightarrow E(-w) \rightarrow \text{Sym}^2 E \rightarrow O_X \rightarrow 0$, which is derived from (2), we find that $\text{Sym}^2(\tilde{\iota})$ acts as follows on $\text{Sym}^2 E|_p$:

$$\begin{cases} \text{diag}(1, 1, 1) & \text{if } p \neq w \\ \text{diag}(1, -1, 1) & \text{if } p = w \end{cases} \quad \text{or} \quad \begin{cases} \text{diag}(1, -1, 1) & \text{if } p \neq w \\ \text{diag}(1, 1, 1) & \text{if } p = w. \end{cases} \quad (4)$$

Suppose the first possibility occurs. Since ι acts by $z \mapsto -z$ in a neighbourhood of a Weierstrass point, $d\iota|_p = -1$. Thm. 17 then gives

$$h^1(X, \text{Sym}^2 E)_- - h^1(X, \text{Sym}^2 E)_+ = \frac{5 \times 3 + 1 \times 1}{2} = 8$$

which is impossible since $h^1(X, \text{Sym}^2 E)_- + h^1(X, \text{Sym}^2 E)_+ = 6$.

Therefore, the second possibility in (4) occurs, and Thm. 17 gives

$$h^1(X, \text{Sym}^2 E)_- - h^1(X, \text{Sym}^2 E)_+ = \frac{5 \times 1 + 1 \times 3}{2} = 4.$$

Solving, we obtain $h^1(X, \text{Sym}^2 E)_+ = 1$ and $h^1(X, \text{Sym}^2 E)_- = 5$.

By Lemma 14, the kernel of \widehat{D}_E is of dimension at most one, so it must be equal to $H^1(X, \text{Sym}^2 E)_+$.

Thus, the point $\mathbb{P}H^1(X, \text{Sym}^2 E)_+ \in \mathbb{P}_E^5$ defines a stable rank four symplectic bundle without a theta divisor, that is, a base point of $|\Xi|$ in \mathcal{M}_2 .

In the following sections, we will show that in fact every base point of $|\Xi|$ is of this form.

7 Maximal Lagrangian subbundles

We write E_e for the extension of O_X by $O_X(-w)$ defined by

$$e \in \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-w))) = \mathbb{P}^1.$$

We denote W the base point of $|\Xi|$ constructed from E_e in the last section.

Lemma 18 *The isomorphism class of the bundle W is independent of the extension class e .*

Proof

We show firstly that W contains a subbundle of the form E_f for every $f \in \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-w)))$.

Since we have homomorphisms $\beta: E_f \rightarrow O_X$ and $\gamma: O_X \rightarrow E_e^*$, we can find a map $\gamma \circ \beta: E_f \rightarrow E_e^*$ for every f . Since $h^1(X, \text{Hom}(E_f, O_X)) = 1$, the map β must be equivariant or antiequivariant; examining the action of the linearisation induced on $\text{Hom}(E_f, O_X)$ by those on E_f and O_X , we find that it is invariant. We check similarly that γ is equivariant.

We show that $\gamma \circ \beta$ factorises via W . By Narasimhan–Ramanan [25], Lemma 3.3, this happens if and only if $\delta(W)$ lies in the kernel of the map $(\gamma \circ \beta)^*: H^1(X, \text{Sym}^2 E_e) \rightarrow H^1(X, \text{Hom}(E_f, E_e))$. This map factorises as

$$H^1(X, \text{Sym}^2 E_e) \xrightarrow{\gamma^*} H^1(X, \text{Hom}(O_X, E_e)) \xrightarrow{\beta^*} H^1(X, \text{Hom}(E_f, E_e)).$$

We see that $\gamma^*(\delta(W))$ is not zero because then, by the same result, O_X would be a subbundle of W , which is excluded by Lemma 10. Now the space $H^1(X, \text{Hom}(O_X, E_e))$ also has an action of ι , for which γ^* is equivariant. Thus it will be enough to show that $\beta^*(H^1(X, \text{Hom}(O_X, E_e))_+)$ is zero.

Taking $\text{Hom}(-, E_e)$ of the sequence $O \rightarrow O_X(-w) \rightarrow E_f \xrightarrow{\beta} O_X \rightarrow 0$, we find that β^* fits into the exact sequence

$$\begin{aligned} \cdots \rightarrow H^1(X, \text{Hom}(O_X, E_e)) &\xrightarrow{\beta^*} H^1(X, \text{Hom}(E_f, E_e)) \rightarrow \\ &H^1(X, \text{Hom}(O_X(-w), E_e)) \rightarrow 0. \end{aligned}$$

Now all the maps on cohomology are ι -equivariant, so this sequence splits into a direct sum of invariant and antiinvariant sequences. We calculate the numbers $h^1(X, \text{Hom}(E_f, E_e))_+$ and $h^1(X, \text{Hom}(O_X(-w), E_e))_+$. We can assume $e \neq f$, so $h^0(X, \text{Hom}(E_f, E_e)) = 0$ by stability, and so

$$h^1(X, \text{Hom}(E_f, E_e))_+ + h^1(X, \text{Hom}(E_f, E_e))_- = 4$$

by Riemann–Roch. Locally, $\text{Hom}(E_f, E_e)$ splits into

$$\begin{aligned} \text{Hom}(O_X(-w), O_X(-w)) \oplus \text{Hom}(O_X, O_X(-w)) \\ \oplus \text{Hom}(O_X(-w), O_X) \oplus \text{Hom}(O_X, O_X) \end{aligned}$$

so, with linearisation as on E in the last section, the action of a lifting of ι to $\text{Hom}(E_f, E_e)$ on the fibre over a Weierstrass point p is given by

$$\begin{cases} \text{diag}(1, -1, -1, 1) & \text{if } p \neq w \\ \text{diag}(1, 1, 1, 1) & \text{if } p = w. \end{cases}$$

Then Theorem 17 gives us $h^1(X, \text{Hom}(E_f, E_e))_- - h^1(X, \text{Hom}(E_f, E_e))_+ = 2$, so $h^1(X, \text{Hom}(E_f, E_e))_- = 3$ and $h^1(X, \text{Hom}(E_f, E_e))_+ = 1$.

As for $h^1(X, \text{Hom}(O_X(-w), E_e))_+$: the bundle $\text{Hom}(O_X(-w), E_e)$ has global sections, of which we must take account when using Theorem 17. There is a single independent section $O_X(-w) \rightarrow E_e$ by the second statement of Lemma 12, which, as before, is equivariant. Taking this into account, we calculate, using Theorem 17 as before, that

$$h^1(X, \text{Hom}(O_X(-w), E_e))_- - h^1(X, \text{Hom}(O_X(-w), E_e))_+ = 0.$$

Then by Riemann–Roch we have $h^1(X, \text{Hom}(O_X(-w), E_e)) = 2$, and hence $h^1(X, \text{Hom}(O_X(-w), E_e))_+ = 1$.

Putting all this together, we have an exact sequence of vector spaces

$$\begin{aligned} H^1(X, \text{Hom}(O_X, E_e))_+ &\xrightarrow{\beta^*} H^1(X, \text{Hom}(E_e, E_f))_+ \\ &\rightarrow H^1(X, \text{Hom}(O_X(-w), E_e))_+ \rightarrow 0. \end{aligned}$$

where each of the last two spaces is of dimension one. Thus, in particular $\beta^*(H^1(X, \text{Hom}(O_X, E_e))_+) = 0$, so E_f belongs to W , as we wanted.

Now we have a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & O_X(-w) & \longrightarrow & E_f & \longrightarrow & O_X \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_e & \xrightarrow{\beta} & W & \xrightarrow{t\beta\circ\omega^{-1}} & E_e^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow t\gamma \\ 0 & \longrightarrow & O_X & \longrightarrow & F & \longrightarrow & O_X(w) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where $\omega: W \rightarrow W^*$ is an isomorphism, unique up to scalar. The only maps whose existences are not immediate are those $O_X \rightarrow F$ and $F \rightarrow O_X(w)$. For the first: since $E_e \cap E_f$ is the subbundle $O_X(-w)$ of W , the image of E_e in F is $E_e/O_X(-w)$, which is O_X . For the second, we note that $\det(F) = \det(W) \det(E_f)^{-1} = O_X(w)$, so F/O_X is isomorphic to $O_X(w)$. By the second statement of Lemma 12, there is only one independent map each from $F \rightarrow O_X(w)$ and $E_e^* \rightarrow O_X(w)$, so the bottom square is commutative after multiplication of the map $F \rightarrow O_X(w)$ by a scalar.

Proposition 19 *The bundle F is isomorphic to E_f^* .*

Proof

Firstly, note that F is not a split extension since then we would have a nonzero map $W \rightarrow O_X$, contradicting stability.

Since the map $\beta \circ \gamma: O_X(-w) \rightarrow W$ factorises via E_f , the class $\delta(E_f)$ belongs to the kernel of the induced map

$$(\beta \circ \gamma)_* : H^1(X, \text{Hom}(O_X, O_X(-w))) \rightarrow H^1(X, \text{Hom}(O_X, W)).$$

Similarly, since $({}^t\gamma \circ {}^t\beta \circ \omega^{-1}) : W \rightarrow O_X(w)$ factorises via F , the class $\delta(F)$ belongs to the kernel of the induced map

$$({}^t\gamma \circ {}^t\beta \circ \omega^{-1})^* : H^1(X, \text{Hom}(O_X(w), O_X)) \rightarrow H^1(X, \text{Hom}(W, O_X)).$$

Now there is a commutative diagram

$$\begin{array}{ccc} H^1(X, \text{Hom}(O_X, O_X(-w))) & \xrightarrow{\sim} & H^1(X, \text{Hom}(O_X(w), O_X)) \\ \gamma_* \downarrow & & \downarrow {}^t\gamma^* \\ H^1(X, \text{Hom}(O_X, E_e)) & \xrightarrow{\sim} & H^1(X, \text{Hom}(E_e^*, O_X)) \\ \beta_* \downarrow & & \downarrow ({}^t\beta \circ \omega^{-1})^* \\ H^1(X, \text{Hom}(O_X, W)) & \xrightarrow{\sim} & H^1(X, \text{Hom}(W, O_X)) \end{array}$$

where the horizontal arrows are induced by the transposes (the lowest one factorises as

$$\begin{aligned} H^1(X, \text{Hom}(O_X, W)) & \xrightarrow{\text{transpose}} H^1(X, \text{Hom}(W^*, O_X)) \\ & \xrightarrow{(\omega^{-1})^*} H^1(X, \text{Hom}(W, O_X)) \quad). \end{aligned}$$

We have seen that $\delta(E_f)$ and $\delta(F)$ belong to the kernels of the composed vertical maps. Since these kernels are each of dimension 1, we see that $\delta(F)$

is proportional to ${}^t(-\delta(E_f)) = \delta(E_f^*)$, whence the proposition. \square

The last step is to show that the class $\delta_f(W) \in H^1(X, \text{Hom}(E_f^*, E_f))$ is symmetric. We recall that the involution $\delta \mapsto -{}^t\delta$ on the extension space $H^1(X, \text{Hom}(E_f^*, E_f))$ sends the class of an extension V to that of V^* .

Now by Lemma 20 from the Appendix (see also the proof of Lemma 23), we have $h^0(X, W(x)) = 1$ for all $x \in X$. A generic E_f has two line subbundles of the form $O_X(-x)$ and $O_X(-\iota x)$ for some $x \in X$, and these generate E_f . Therefore we have $h^0(X, \text{Hom}(E_f, W)) = 1$ for such an E_f . Since $W \cong W^*$, we deduce that there is an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_f & \longrightarrow & W & \longrightarrow & E_f^* \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & E_f & \longrightarrow & W^* & \longrightarrow & E_f^* \longrightarrow 0 \end{array}$$

with classes $\delta_f(W)$ and $-{}^t\delta_f(W)$. Since the bundles E and E^* are simple, these two classes are proportional, so $\delta_f(W)$ belongs to either $H^1(X, \text{Sym}^2 E_f)$ or $H^1(X, \bigwedge^2 E_f)$. If it were the latter then W would have an orthogonal structure by Theorem 4, which would contradict the stability of W . Thus $\delta_f(W) \in H^1(X, \text{Sym}^2 E_f)$ and W is the base point of $|\Xi|$ associated to E_f , which is unique by Lemma 14. Since the symmetry of $\delta_f(W)$ is a closed condition on f , we see that $\delta_f(W)$ is symmetric for all f .

This completes the proof of Lemma 18. \square

In this way we associate to each Weierstrass point $w \in X$ a base point of $|\Xi|$. We denote this bundle W_w .

8 Characterisation of the base points

In this section we will prove Theorem 3. We will show that an arbitrary $W \in \mathcal{M}_2$ with no theta divisor must be of the form W_w for some Weierstrass point $w \in X$, and that for distinct $w, v \in X$, the bundles W_w and W_v are mutually nonisomorphic.

Let $W \in \mathcal{M}_2$, then, be a base point of $|\Xi|$. Since $S(W) = J^1$, in particular we have $h^0(X, W(x)) > 0$ for all $x \in X$. Now by Riemann–Roch, we have $\chi(X, W \otimes K_X) = 4$. By Serre duality $h^1(X, W \otimes K_X) = h^0(X, W^*)$, which is zero since W is stable by Prop. 10. Thus $h^0(X, W \otimes K_X) = 4$. On the other

hand,

$$h^0(X, W \otimes K_X(-x)) = h^0(X, \text{Hom}(O_X(-\iota x), W)) > 0$$

for all $x \in X$, by hypothesis. This means that the rank 4 bundle $W \otimes K_X$ is not generated by its global sections. In other words, the evaluation map

$$\text{ev}: O_X \otimes H^0(X, W \otimes K_X) \rightarrow W \otimes K_X$$

is not of maximal rank. We denote \mathcal{F} the subsheaf of \mathcal{W} corresponding to the image of ev .

Lemma 20 *The subsheaf \mathcal{F} of $\mathcal{W} \otimes K_X$ corresponds to a vector subbundle $F \subset W \otimes K_X$ which has rank three and degree five, and is stable.*

Proof

This proof is straightforward but rather long. We relegate it to the appendix, in order not to interrupt the story.

By for example taking global sections of $0 \rightarrow F \rightarrow W \otimes K_X \rightarrow \cdots$, we have $H^0(X, F) \cong H^0(X, W \otimes K_X)$. Denote $L^{-1} = \text{Ker}(\text{ev})$. There is an exact sequence

$$0 \rightarrow L^{-1} \rightarrow O_X \otimes H^0(X, F) \rightarrow F \rightarrow 0. \quad (5)$$

Henceforth we denote F by F_L to emphasise that F depends on L . Note that $\det(F_L) = L$.

Twisting by K_X^{-1} , we have shown that every $W \in \mathcal{M}_2$ with no theta divisor contains a rank three subbundle of the form $F_L \otimes K_X^{-1}$, for some $L \in J_X^5$. We will now say more about the structure of F_L .

Proposition 21 *There is an exact sequence*

$$0 \rightarrow K_X^{-2}L \rightarrow F_L \rightarrow K_X \oplus K_X \rightarrow 0.$$

Proof

Let us show firstly that $h^0(X, \text{Hom}(F_L, K_X)) = 2$. We have an exact sequence

$$0 \rightarrow H^0(X, F_L^* \otimes K_X) \rightarrow H^0(X, K_X) \otimes H^0(X, L) \xrightarrow{\mu} H^0(X, K_X L) \rightarrow \cdots$$

given by tensoring the dual of (5) by K_X and taking global sections. Thus $H^0(X, F_L^* \otimes K_X) = \text{Ker}(\mu)$. Now since X is of genus two, it is not hard to see that there is a short exact sequence

$$0 \rightarrow K_X^{-1} \rightarrow O_X \otimes H^0(X, K_X) \rightarrow K_X \rightarrow 0.$$

Tensoring by L and taking cohomology, we obtain

$$0 \rightarrow H^0(X, K_X^{-1}L) \rightarrow H^0(X, K_X) \otimes H^0(X, L) \xrightarrow{\mu} H^0(X, K_X L) \rightarrow \dots$$

whence $\text{Ker}(\mu) = H^0(X, K_X^{-1}L)$, which is of dimension two by Riemann–Roch.

We choose a basis u, v for $H^0(X, F_L^* \otimes K_X)$. This gives a map

$$(u, v): F_L \rightarrow K_X \oplus K_X.$$

This is of maximal rank: if (u, v) factorised via a line bundle M we would have $M = K_X$, but u and v were chosen to be linearly independent. Now (u, v) maps F_L surjectively to a rank two subsheaf \mathcal{G} of $\mathcal{K}_X \oplus \mathcal{K}_X$. Since F_L is stable, $\mu(\mathcal{G}) > 5/3$, and since \mathcal{G} is a subsheaf of $\mathcal{K}_X \oplus \mathcal{K}_X$, we have $\mu(\mathcal{G}) \leq 2$. Since $\mu(\mathcal{G})$ is a half integer between $5/3$ and 2 , it is equal to 2 and (u, v) is surjective.

Taking determinants, we find $\text{Ker}(u, v) \cong K_X^{-2}L$. Thus we have the sequence $0 \rightarrow K_X^{-2}L \rightarrow F_L \rightarrow K_X \oplus K_X \rightarrow 0$. This completes the proof of Prop. 21. \square

We now show that, up to scalar, the above map $K_X^{-2}L \rightarrow F_L$ is unique.

Proposition 22 *The bundle $F_L \otimes K_X^2 L^{-1}$ has just one independent section.*

Proof

Tensoring (5) by $K_X^2 L^{-1}$ and taking global sections, we obtain

$$0 \rightarrow H^0(X, F_L \otimes K_X^2 L^{-1}) \rightarrow H^1(X, K_X^2 L^{-2}) \rightarrow H^1(X, K_X^2 L^{-1}) \otimes H^0(X, L)^*$$

and one checks that the second map is identified with the cup product map

$$H^1(X, K_X^2 L^{-2}) \rightarrow \text{Hom}(H^0(X, L), H^1(X, K_X^2 L^{-1})).$$

By Lemma 6, then, $H^0(X, F_L \otimes K_X^2 L^{-1})^*$ appears as the cokernel of the multiplication map $\mu: H^0(X, K_X^{-1}L) \otimes H^0(X, L) \rightarrow H^0(X, K_X^{-1}L^2) = \mathbb{C}^7$. To determine this cokernel, we consider two cases:

(a) Suppose $|K_X^{-1}L|$ is base point free. In this case there is an exact sequence of vector bundles $0 \rightarrow K_X L^{-1} \rightarrow \mathcal{O}_X \otimes H^0(X, K_X^{-1}L) \rightarrow K_X^{-1}L \rightarrow 0$, the surjection being the evaluation map. Tensoring by L and taking global sections, we obtain

$$\begin{aligned} 0 \rightarrow H^0(X, K_X) \rightarrow H^0(X, L) \otimes H^0(X, K_X^{-1}L) &\xrightarrow{\mu} \\ &H^0(X, K_X^{-1}L^2) \rightarrow H^1(X, K_X) \rightarrow 0 \end{aligned}$$

whence $\text{Coker}(\mu) = H^1(X, K_X)$, so is of dimension one.

(b) If $K_X^{-1}L = K_X(x)$ for some $x \in X$ then $L = K_X^2(x)$ and $H^0(X, K_X^{-1}L) = H^0(X, K_X(x))$. We have a diagram

$$\begin{array}{ccc} H^0(X, K_X^{-1}L) \otimes H^0(X, L) & \xrightarrow{\mu} & H^0(X, K_X^{-1}L^2) \\ \downarrow \parallel & & \downarrow \parallel \\ H^0(X, K_X(x)) \otimes H^0(X, K_X^2(x)) & \xrightarrow{\mu} & H^0(X, K_X^3(2x)) \end{array}$$

By the base point free pencil trick (Arbarello et al [1], p. 126), the kernel of μ is isomorphic to $H^0(X, K_X(x)) = \mathbb{C}^2$. Now $h^0(X, K_X(x)) \cdot h^0(X, K_X^2(x)) = 2 \times 4 = 8$, so the image of μ is of dimension six. Thus $\text{Coker}(\mu)$ is again of dimension one.

In either case, we obtain $h^0(X, F_L \otimes K_X^2 L^{-1}) = 1$. \square

Now the symplectic form on W induces a symplectic form on $W \otimes K_X$ with values in K_X^2 . Furthermore, it is not hard to show (using for example Theorem 4) that a subbundle $G \subset W$ is isotropic in W if and only if $G \otimes K_X$ is isotropic in $W \otimes K_X$.

Lemma 23 *The degree one line bundle $K_X^{-2}L$ is effective.*

Proof

Firstly, we claim that $h^0(X, W(x)) = 1$ for all $x \in X$. To see this, note that $H^0(X, W(x)) \cong H^0(X, W \otimes K_X(-\iota x))$ and the latter space is identified with $\text{Ker}(O_X \otimes H^0(X, W \otimes K_X)|_{\iota x} \rightarrow F_L|_{\iota x})$. This map is surjective by Lemma 20, so the kernel is of dimension one. This also shows that the generic value of $h^0(X, L \otimes W)$ as L ranges over J^1 is 1.

By Thm. 8, then, for each $x \in X$ which is not a Weierstrass point, the line subbundles $K_X(-x) = O_X(\iota x)$ and $K_X(-\iota x) = O_X(x)$ define a rank two subsheaf of $W \otimes K_X$, generating a rank two subbundle G_x of $W \otimes K_X$ on which the symplectic form inherited from $W \otimes K_X$ vanishes identically⁵.

We can say even more about the Lagrangian subbundles G_x . The line subbundles $O_X(x) \subset W \otimes K_X$ are contained in F_L , because an inclusion $O_X(x) \hookrightarrow W \otimes K_X$ is equivalent to a section $O_X \rightarrow W \otimes K_X$ vanishing at x , and all sections of $W \otimes K_X$ are by definition F_L -valued. Therefore G_x also belongs to F_L .

We now give a more geometric way to realise the restriction of the K_X^2 -valued symplectic form on $W \otimes K_X$ to F_L . Since F_L has rank greater than

⁵It is not hard to show, using for example Theorem 4, that a subbundle $G \subset W$ is Lagrangian if and only if $G \otimes K_X$ is Lagrangian with respect to the K_X^2 -valued symplectic form on $W \otimes K_X$.

two, the form does not restrict to zero (although it is degenerate).

Claim: $h^0(X, \text{Hom}(\bigwedge^2 F_L, K_X^2)) = 1$. We have $\bigwedge^2 F_L \cong \text{Hom}(F_L, \det(F_L))$ since F_L is of rank three. Therefore $\bigwedge^2 F_L^* \otimes K_X^2 \cong F_L \otimes K_X^2 L^{-1}$, and this bundle has only one global section by Prop. 22.

Let α be a map $K_X^{-2}L \rightarrow F_L$. Then

$$w_1 \wedge w_2 \mapsto w_1 \wedge w_2 \wedge \alpha(\cdot) \quad (6)$$

defines a map $\bigwedge^2 F_L \rightarrow \text{Hom}(K_X^{-2}L, \det F_L) \cong K_X^2$. Clearly this is nonzero and, by the Claim, must be a scalar multiple of the restricted symplectic form.

Therefore, the isotropy of G_x in $W \otimes K_X$ implies that the map

$$(O_X(\iota x) \oplus O_X(x) \oplus K_X^{-2}L) \rightarrow F_L$$

is not of maximal rank. In particular, for *every* $x \in X$ apart from the Weierstrass points, the line subbundle $K_X^{-2}L \subset F_L$ belongs to G_x .

Since F_L is stable, G_x has degree two or three. If it is two then in fact $G_x = O_X(\iota x) \oplus O_X(x)$. Now $K_X^{-2}L$ is also a degree one subbundle of G_x , so is equal to $O_X(x)$ or $O_X(\iota x)$.

If G_x has degree three then it is an elementary transformation

$$0 \rightarrow \mathcal{O}_X(x) \oplus \mathcal{O}_X(\iota x) \rightarrow \mathcal{G}_x \rightarrow \mathbb{C}_a \rightarrow 0$$

for some $a \in X$. The determinant of G_x is therefore $K_X(a)$. Since $K_X^{-2}L$ belongs to G_x , there is a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_X^{-2}L & \longrightarrow & G_x & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_X^{-2}L & \longrightarrow & F_L & \longrightarrow & K_X \oplus K_X \longrightarrow 0 \end{array}$$

where M is a line bundle of degree two; clearly M is isomorphic to K_X . Therefore, since $\det(G_x) = K_X(a)$, the bundle $K_X^{-2}L$ is isomorphic to $O_X(a)$. In either case, $K_X^{-2}L$ is effective. \square

Note that $K^{-2}L \otimes K_X^{-1} = O_X(-\iota a)$. By Lemma 21, the bundle $F_L \otimes K_X^{-1}$ determines an extension class $e \in H^1(X, \text{Hom}(O_X \oplus O_X, O_X(-\iota a)))$ which is of the form (e_1, e_2) for some $e_1, e_2 \in H^1(X, \text{Hom}(O_X, O_X(-\iota a)))$, a vector space of dimension two. Now we claim that e_1 and e_2 form a basis of this

space. For, if $e_2 = \mu e_1$ for some $\mu \in \mathbb{C}$ then the map $O_X \rightarrow O_X \oplus O_X$ given by $\lambda \mapsto (-\mu\lambda, \lambda)$ would lift to $F_L \otimes K_X^{-1}$, contradicting stability of this bundle.

Now we consider a homomorphism $O_X \rightarrow O_X \oplus O_X$ given by $\lambda \mapsto (\alpha\lambda, \beta\lambda)$. The inverse image of the subbundle $(\alpha, \beta)(O_X) \subset O_X \oplus O_X$ is an extension E of O_X by $O_X(-\iota a)$, and one sees, for example by inspecting transition functions, that the extension class of E is

$$\alpha e_1 + \beta e_2 \in H^1(X, \text{Hom}(O_X, O_X(-\iota a))).$$

Letting $(\alpha : \beta)$ vary in \mathbb{P}^1 , we find that all nontrivial extensions of this form are subbundles of $F_L \otimes K_X^{-1}$ since $\{e_1, e_2\}$ is a basis of the space $H^1(X, \text{Hom}(O_X, O_X(-\iota a)))$. They are also isotropic in W because each one contains a pair of line subbundles of the form

$$O_X(-p), \quad O_X(-\iota p) = K_X^{-1}(p)$$

for some $p \in X$. Since $h^0(X, \text{Hom}(O_X(-x), W)) = 1$ for all $x \in X$, by Thm. 8 these subbundles must generate an isotropic subbundle of rank two.

Lemma 24 *The point a is a Weierstrass point of X .*

Proof

We have just seen that W contains a pencil of isotropic subbundles which are nontrivial extensions $0 \rightarrow O_X(-\iota a) \rightarrow E \rightarrow O_X \rightarrow 0$. It will suffice to show that if a is not a Weierstrass point, then any bundle containing such a pencil must have a theta divisor. To do this, we will show that for one of these E , the image of the map $j_E: J^1 \dashrightarrow (\mathbb{P}_E^5)^*$ considered in section 5 is nondegenerate. By Lemma 13, any symplectic bundle containing this E as an isotropic subbundle must have a theta divisor.

Suppose, then, that $\iota a \neq a$. We must show that one of the extensions E above satisfies both of the genericity hypotheses of Lemma 14. Let $p \in X$ be a point such that $p \neq \iota p$, $p \neq a$ and $p \neq \iota a$ and consider the extension E of O_X by $O_X(-a)$ defined by the divisor

$$a + p + \iota p \in |K_X(a)| = \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-a))).$$

Then E^* fits into a short exact sequence $0 \rightarrow M \rightarrow E^* \rightarrow O_X(p) \rightarrow 0$ where $M := O_X(a - p)$. Firstly, M is not a point of order two in $\text{Pic}^0(X)$. For, otherwise we would have $O_X(2a) = O_X(2p)$, whence either a and p are both Weierstrass points or $p = a$, both of which are excluded by our hypothesis.

This means that there exists a unique pair of points x, y such that $K_X M^2 = O_X(x + y)$. Notice that $2a + \iota p \sim p + x + y$.

It remains to check that E satisfies the other genericity hypothesis of Lemma 14, so that j_E is defined at the points which would map to the point $\mathbb{P}H^0(X, K_X M^2)$ in $(\mathbb{P}_E^5)^*$. We check as before that $j_E(L)$ is this point if and only if $L^{-1} = M(-u)$ and $K_X^{-1}L = M(-v)$, for some $u, v \in X$, and that the only solution to these equations (up to exchanging L and $K_X L^{-1}$) is

$$L = M^{-1}(x) = M(\iota y) \quad \text{and} \quad K_X L^{-1} = M^{-1}(y) = M(\iota x).$$

Now j_E is not defined at $M^{-1}(x)$ if and only if

$$h^0(X, M^{-1}(x) \otimes E^*) \geq 2 \quad \text{and/or} \quad h^0(X, M^{-1}(y) \otimes E^*) \geq 2,$$

equivalently, since $\chi(X, M^{-1}(x) \otimes E^*) = 1 = \chi(X, M^{-1}(y) \otimes E^*)$,

$$h^1(X, M^{-1}(x) \otimes E^*) \geq 1 \quad \text{and/or} \quad h^1(X, M^{-1}(y) \otimes E^*) \geq 1.$$

By Serre duality, this becomes

$$h^0(X, K_X M(-x) \otimes E) \geq 1 \quad \text{and/or} \quad h^0(X, K_X M(-y) \otimes E) \geq 1,$$

that is, $K_X^{-1}M^{-1}(x) = M^{-1}(-\iota x)$ or $K_X^{-1}M^{-1}(y) = M^{-1}(-\iota y)$ belongs to E .

Thus, in order that j_E be defined at L (and thus also at $K_X L^{-1}$), we require that the sets

$$\{M^{-1}(-\iota x), M^{-1}(-\iota y)\} \quad \text{and} \quad \{O_X(-a), O_X(-p), O_X(-\iota p)\}$$

be disjoint. Since $O_X(-\iota p) = K_X^{-1}(p)$, in fact we can replace the second one by $\{O_X(-a), O_X(-p)\}$.

Suppose $M^{-1}(-\iota x) = O_X(-a)$. Then $O_X(p - a - \iota x) = O_X(-a)$, so $O_X(p + a) = O_X(\iota x + a)$. Thus $p = \iota x$. But then $O_X(p + x + y) = K_X(y)$. Since $2a + \iota p \sim p + x + y$, either a is a Weierstrass point or $a = p$, contrary to hypothesis. The other possibilities can be excluded in a similar manner.

Thus j_E is defined at the required points and so its image in $(\mathbb{P}_E^5)^*$ is nondegenerate. By Lemma 13, for such an E , some symplectic extension of E^* by E has no theta divisor only if a is a Weierstrass point. \square

We have shown, therefore, that $F_L \otimes K_X^{-1}$ contains all nontrivial extensions

$$0 \rightarrow O_X(-a) \rightarrow E \rightarrow O_X \rightarrow 0$$

as isotropic subbundles. Since, by Lemma 14, there is at most one symplectic extension of each of E^* by E with no theta divisor, the bundle W is isomorphic to the bundle W_a constructed in section 6 (and we get another proof of the fact that W_a contains all nontrivial extensions E of the above form as

Lagrangian subbundles, so the isomorphism class of W_a depends only on a). Thus there are at most six possibilities for a bundle $W \in \mathcal{M}_2$ with no theta divisor.

It remains to show that the base locus of $|\Xi|$ is reduced.

Lemma 25 *Let $w, v \in X$ be distinct Weierstrass points. Then bundles W_v and W_w are mutually nonisomorphic.*

Proof

Let E be a nontrivial extension of O_X by $O_X(-w)$ defined by a divisor $p + \iota p + w \in |K_X(w)|$ where $p \neq v$. Now W_w is an extension of E^* by E by Lemma 18. Consider now nontrivial extensions of the form

$$0 \rightarrow O_X(-v) \rightarrow F_f \rightarrow O_X \rightarrow 0 \quad (7)$$

parametrised by $f := \langle \delta(F) \rangle \in \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-v))) = \mathbb{P}^1$. If W_v were isomorphic⁶ to W_w then we would have a map $F_f \hookrightarrow W_w$ for all such F_f , again by Lemma 18.

There are three possibilities for the rank of the subsheaf $\mathcal{F}_f \cap \mathcal{E}$: these are zero, one and two. It is never two, because this would imply that some F_f were isomorphic to E , which is impossible since $O_X(-v) \subset F_f$ for all f but $O_X(-v) \not\subset E$. If it is one then it is not hard to check⁷ that the subsheaf $\mathcal{F}_f \cap \mathcal{E}$ corresponds to a line subbundle of degree -1 in F_f and E , but F_f and E have such a subbundle in common for only finitely many f . Thus if $W_v \cong W_w$ then we have maps $F_f \rightarrow E^*$ of generic rank two for almost all $f \in \mathbb{P}H^1(X, \text{Hom}(O_X, O_X(-v)))$.

By Riemann–Roch and since

$$h^1(X, \text{Hom}(O_X(-v), E^*)) = h^0(X, \text{Hom}(O_X(-v), E)) = 0,$$

there is one independent map $\gamma: O_X(-v) \rightarrow E^*$. An extension F_f admits a map to E^* of generic rank two only if γ factorises via F_f . By Narasimhan–Ramanan [25], Lemma 3.2, this is equivalent to $\delta(F_f)$ belonging to the kernel of the map

$$H^1(X, \text{Hom}(O_X, O_X(-v))) \rightarrow H^1(X, \text{Hom}(O_X, E^*))$$

induced by γ . Via Serre duality, this is dual to the map

$$H^0(X, K_X \otimes E) \rightarrow H^0(X, K_X(v)) \quad (8)$$

⁶A priori, we should only ask for S -equivalence but, by Prop. 10, the base locus of $|\Xi|$ consists of stable vector bundles so S -equivalence of W_v and W_w implies isomorphism.

⁷See for example the proof of [13], Prop. 18.

induced by $\gamma^*: E \rightarrow \mathcal{O}_X(v)$. We have rank two maps $F_f \rightarrow E^*$ for almost all F_f if and only if (8) is zero. Now there is an exact sheaf sequence

$$\mathcal{N} \rightarrow \mathcal{E} \xrightarrow{\gamma^*} \mathcal{O}_X(v)$$

where \mathcal{N} is invertible. The map in (8) is zero only if every map $K_X^{-1} \rightarrow E$ factorises via the line subbundle $N' \subset E$ generated by \mathcal{N} ; this has degree at most -1 . Now $\chi(X, \text{Hom}(K_X^{-1}, E)) = 1$ and

$$h^1(X, \text{Hom}(K_X^{-1}, E)) = h^0(X, E^*) = 1,$$

so $h^0(X, \text{Hom}(K_X^{-1}, E)) = 2$. But $\text{Hom}(K_X^{-1}N')$ is a line bundle of degree at most one, so there is at most one independent map $K_X^{-1} \rightarrow N'$. Therefore not every map $K_X^{-1} \rightarrow E$ can factorise via $N' \hookrightarrow E$.

This means that there are no rank two maps $F_f \rightarrow E^*$ for most extensions F_f . Putting all this together, there cannot be maps $F_f \rightarrow W_w$ for all F_f . Hence W_v cannot be isomorphic to W_w . \square

In summary, every symplectic bundle of rank four over X with no theta divisor is of the form W_w for some Weierstrass point w , and the bundles W_w and W_v are mutually nonisomorphic for $w \neq v$.

This completes the proof of Theorem 3, our main result. \square

9 The link with Raynaud bundles

In [33], Raynaud gives examples of semistable bundles without theta divisors over curves of arbitrary genus. Arnaud Beauville has shown that in the rank four / genus two case, some of these bundles admit symplectic or orthogonal structures. In this section, we show how the extensions in § 6 of the present article are related to Raynaud and Beauville's work.

9.1 Raynaud's construction in genus 2

This subsection is expository; the reference is Raynaud [33], sect. 3.

We write $J := J^0$ for brevity and identify J with its dual Abelian variety $\widehat{J} = \text{Pic}^0(J)$ by means of the principal polarisation. We choose a symmetric divisor on J defining the principal polarisation and, abusing notation, denote it Θ . Consider the (ample) bundle $\mathcal{O}_J(2\Theta)$ on J . Via our identification $J \xrightarrow{\sim} \widehat{J}$, the map $\phi_{2\Theta}: J \rightarrow \widehat{J}$ is identified with the duplication map

$2_J: J \rightarrow J$, which has degree $2^{2g} = 16$.

Let \mathbf{P} be the Poincaré bundle over $J \times J$ which is trivial over $\{0\} \times J$ and $J \times \{0\}$. We write p and q for the projections of $J \times J$ to the first and second factors.

Recall that a sheaf N over J is *WIT* (“Weak Index Theorem”) of index i if the sheaves $R^j q_*(p^* N \otimes \mathbf{P})$ are zero for all $j \neq i$. Following Birkenhake–Lange [9], p. 445, we define the *Fourier–Mukai transform* of such an N as $R^i q_*(p^* N \otimes \mathbf{P}) =: F(N)$. Now $\mathcal{O}_J(-2\Theta)$ is WIT of index $g = 2$. In [23], Mukai proved:

- (i) $F(\mathcal{O}_J(-2\Theta))$ is a vector bundle M of rank $(2\Theta)^g/g = 4$ over J .
- (ii) $F(M) \cong (-1_J)^* \mathcal{O}_J(-2\Theta)$.
- (iii) $2_J^*(M) \cong \mathcal{O}_J(2\Theta)^{\oplus 4}$.

Let $U \subset J$ be an open set over which $\mathcal{O}_J(-2\Theta)$ is trivial. Shrinking U if necessary, we can assume that there is a section s of $p^* M \otimes \mathbf{P}$ over $q^{-1}(U) = J \times U$ whose restriction to $\{0\} \times U$ is nonzero. Consider the Abel–Jacobi map $\alpha_c: X \hookrightarrow J$ which sends the point $x \in X$ to $\mathcal{O}_X(x - c) \in J$. We denote the image curve X_c ; clearly this passes through 0. Then $\alpha_c^* M$ is a vector bundle $E_c \rightarrow X$ of rank four. By construction, the restriction of s to $X_c \times U$ is nonzero, whence $h^0(X, E_c \otimes L) > 0$ for generic (and hence all) $L \in J$.

To see that E_c is semistable, we introduce the (possibly nonreduced) curve $Y := 2_J^{-1}(X_c) \subset J$. Write $f: Y \rightarrow X_c$ for the restriction of 2_J to Y , which is a degree 16 map of curves. Now for any vector bundle $V \rightarrow X_c$, we have

$$\mu(f^* V) = \deg(f) \cdot \mu(V) = 16 \mu(V). \quad (9)$$

By (iii), the pullback of E_c to Y is a direct sum of line bundles of the same degree, so is semistable. From this and (9) we check that E_c is also semistable.

Let us find the slope of E_c . Firstly, we have

$$\deg(E_c) = \frac{\deg((2_J^* M)|_Y)}{\deg(f)}.$$

On the other hand, since $(2_J)^* M \cong \mathcal{O}_J(2\Theta)^{\oplus 4}$, we have

$$\begin{aligned} \deg((2_J^* M)|_Y) &= 4 \deg(\mathcal{O}_J(2\Theta)|_Y) \\ &= \deg(\mathcal{O}_J(2\Theta)|_Y^{\otimes 4}) \\ &= \deg(((2_J)^* \mathcal{O}_J(2\Theta))|_Y) \end{aligned}$$

since $(n_J)^*O_J(2\Theta) = O_J(2n^2\Theta)$. This is in turn equal to

$$\deg(2_J) \cdot (\deg(O_J(2\Theta)|_{X_c}) = 16(2\Theta \cdot X_c) = 16 \cdot 2g = 64,$$

so f^*E_c has slope 16. By (9), the slope of E_c is 1.

Thus, the tensor product of E_c by any line bundle of degree -1 is a semistable vector bundle of degree zero and rank four with no theta divisor.

Remark: In fact this is a stable bundle, since, by Raynaud [33], Cor. 1.7.4, all semistable bundles of degree zero and rank at most three over a curve of genus two have a theta divisor.

9.2 Symplectic and orthogonal structures

Here we show that for certain $c \in X$, the bundle E_c admits a symplectic structure.

Note: This construction is due to Arnaud Beauville, to whom I express my thanks for allowing me to present his results here.

We begin by describing three actions of the Heisenberg group $\mathcal{G}(2\Theta)$. Set-theoretically, this group consists of pairs (ϕ, η) where η is a point of order two in J and ϕ an automorphism of the variety $O_J(2\Theta)$ covering t_η ; in other words, such that there is a commutative diagram of varieties

$$\begin{array}{ccc} O_J(2\Theta) & \xrightarrow{\phi} & O_J(2\Theta) \\ \downarrow & & \downarrow \\ J & \xrightarrow{t_\eta} & J. \end{array}$$

Write $J[2] := \text{Ker}(2_J)$. The group $\mathcal{G}(2\Theta)$ is a central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \mathcal{G}(2\Theta) \rightarrow J[2] \rightarrow 0$$

by for example Birkenhake–Lange [9], Prop. 6.1.1 and since $\phi_{n\Theta}$ is multiplication by n on J .

(i) The first action of $\mathcal{G}(2\Theta)$ is on $H^0(J, 2\Theta)$ and is given in [9], § 6.4. For a global section s of $O_J(2\Theta)$, we define $(\phi, \eta) \cdot s = \phi \circ s \circ t_{-\eta}$, which is equal to $\phi \circ s \circ t_\eta$ since $-\eta = \eta$. This gives a representation of $\mathcal{G}(2\Theta)$ in $\text{GL}(H^0(J, O_J(2\Theta)))$.

(ii) There is a natural action of $\mathcal{G}(2\Theta)$ on the total space of $O_J(-2\Theta)$, given by $(\phi, \eta) \cdot k = ({}^t\phi^{-1})(k)$.

(iii) The last action will define a linearisation of $J[2]$ on $O_J(4\Theta)$, which will depend on the choice of a theta characteristic κ . To each κ , we associate a character of weight 2 of $\mathcal{G}(2\Theta)$, denoted χ_κ (see Beauville [3], Lemme A.4). Given $(\phi, \eta) \in \mathcal{G}(2\Theta)$, the map

$$(\phi \otimes \phi) \cdot \chi_\kappa(\phi, \eta)^{-1} \quad (10)$$

is an automorphism of $O_J(4\Theta)$ covering t_η . The subgroup $\mathbb{C}^* \subset \mathcal{G}(2\Theta)$ acts trivially, so this action factorises via $\mathcal{G}(2\Theta) \rightarrow J[2]$.

Fix a theta characteristic κ , and consider the product of these three actions on the bundle

$$Q := \text{Hom} \left(\mathcal{O}_J(2\Theta) \otimes H^0(J, 2\Theta)^*, (\mathcal{O}_J(2\Theta) \otimes H^0(J, 2\Theta)^*)^* \otimes \mathcal{O}_J(4\Theta) \right).$$

We notice firstly that the actions of \mathbb{C}^* on each copy of $O_J(-2\Theta) \otimes H^0(J, 2\Theta)$ cancel. Since the subgroup \mathbb{C}^* acts trivially on $O_J(4\Theta)$, the action of $\mathcal{G}(2\Theta)$ on Q factorises via $\mathcal{G}(2\Theta) \rightarrow J[2]$.

Now by Schneider [36], Prop. 2.1, there is an isomorphism of $J[2]$ -bundles between $O_J(2\Theta) \otimes H^0(J, O_J(2\Theta))^*$ and 2_J^*M , so $O_J(2\Theta) \otimes H^0(J, O_J(2\Theta))^*$ descends to M . Furthermore, we have

Proposition 26 *The quotient of $O_J(4\Theta)$ by the linearisation of the action of $J[2]$ corresponding to the theta characteristic κ is $O_J(\Theta_\kappa)$.*

Proof

The line bundles over J which pull back to $O_J(4\Theta)$ are exactly the $O_J(\Theta_\kappa)$ by for example Birkenhake–Lange, p. 34, and since $\text{Ker}(2_J^*) \cong J[2]$. Therefore, the quotient, if it exists, is among the $O_J(\Theta_\kappa)$.

For each $(\phi, \eta) \in \mathcal{G}(2\Theta)$, the map $\phi^{\otimes 2}$ is an automorphism of $O_J(4\Theta)$ covering t_η . Thus we get a homomorphism $\mathcal{G}(2\Theta) \rightarrow \mathcal{G}(4\Theta)$ and an action of $\mathcal{G}(2\Theta)$ on $H^0(J, O_J(4\Theta))$.

Now each $O_J(\Theta_\kappa)$ has a unique global section up to scalar since the polarisation is principal. We write ϑ_κ for this section. By Beauville [3], Prop. A.8, the pullbacks of the ϑ_κ by 2_J form a basis for the space $H^0(J, O_J(4\Theta))$. Furthermore, for each κ , the section $2_J^*\vartheta_\kappa$ is an eigenvector for the action of $\mathcal{G}(2\Theta)$ via χ_κ ; precisely,

$$(\phi, \eta) \cdot (2_J^*\vartheta_\kappa) = \chi_\kappa(\phi, \eta)(2_J^*\vartheta_\kappa).$$

Now we will fix a theta characteristic κ and twist this action of $\mathcal{G}(2\Theta)$ on $O_J(4\Theta)$ by χ_κ^{-1} , so we get the linearisation of $J[2]$ on $O_J(4\Theta)$ in (iii) above. Since the underlying action of $J[2]$ on J is free, by Kempf's Descent Lemma $O_J(4\Theta)$ can also be descended to $J \cong J/J[2]$ by this linearisation. The induced action of $J[2]$ on $H^0(J, 4\Theta)$ is given by

$$(\phi, \eta) \cdot s = \chi_\kappa(\phi, \eta)^{-1} (\phi^{\otimes 2} \circ s \circ t_\eta).$$

Putting all this together, we see that $2_J^* \vartheta_\kappa$ is equivariant for the linearisation corresponding to κ . Thus it descends to the quotient by this linearisation. It is not hard to see that the descended section vanishes exactly along Θ_κ and that the quotient is $O_J(\Theta_\kappa)$. \square

Thus, for a theta characteristic κ , the bundle Q descends via the linearisation corresponding to κ to $\text{Hom}(M, M^* \otimes O_J(\Theta_\kappa))$. We now describe a section of Q which is equivariant for this linearisation. By Beauville [3], Prop. A.5, there is a basis of

$$\text{Hom}_{\mathbb{C}}(H^0(J, 2\Theta)^*, H^0(J, 2\Theta)) = H^0(J, 2\Theta) \otimes H^0(J, 2\Theta)$$

indexed by the theta characteristics of X . An element of this basis is an eigenvector ξ_κ with respect to the character χ_κ of weight 2:

$$(\phi, \eta) \cdot \xi_\kappa = \chi_\kappa(\phi, \eta) \xi_\kappa. \quad (11)$$

Clearly the bundle Q is isomorphic to $O_J \otimes H^0(J, 2\Theta)^{\otimes 2}$, so each ξ_κ defines a section $1 \otimes \xi_\kappa$ of Q . Let $j \in J$; then one checks that an automorphism $(\phi, \eta) \in \mathcal{G}(2\Theta)$ sends the element $1_j \otimes \xi_\kappa \in O_J|_j \otimes H^0(J, 2\Theta)^{\otimes 2} = Q|_j$ to

$$(\chi_\kappa(\phi, \eta)^{-1} \cdot 1_{j+\eta}) \otimes (\chi_\kappa(\phi, \eta) \xi_\kappa) = 1_{j+\eta} \otimes \xi_\kappa \in Q|_{j+\eta}.$$

Thus $1 \otimes \xi_\kappa$ is an equivariant section for the linearisation of $J[2]$ on Q corresponding to κ . Hence it descends to $\text{Hom}(M, M^* \otimes O_J(\Theta_\kappa))$. Moreover, ξ_κ defines an isomorphism $H^0(J, 2\Theta)^* \rightarrow H^0(J, 2\Theta)$ by [3], Remarque A.6, so $1 \otimes \xi_\kappa$ defines an isomorphism $M \xrightarrow{\sim} M^* \otimes O_J(\Theta_\kappa)$. Furthermore, by [3], Prop. A.5, this isomorphism is symmetric or antisymmetric according to the parity of κ . In this way we get a symplectic (resp., orthogonal) structure on M for each odd (resp., even) theta characteristic.

Now let $w \in X$ be a Weierstrass point and consider the Abel–Jacobi map $\alpha_w: X \rightarrow J$. We write $\tau := O_X(w)$. Denote E_w the bundle $\alpha_w^* M \rightarrow X$. By the above argument, there exists an antisymmetric isomorphism

$$1 \otimes \xi_\tau: E_w \rightarrow E_w^* \otimes (O_J(\Theta_\tau)|_{X_w}).$$

Now K_J is trivial and $X_w = \Theta_\tau$ as divisors. Therefore, by the adjunction formula,

$$O_J(\Theta_\tau)|_{X_w} \cong K_J \otimes O_J(X_w)|_{X_w} \cong K_X,$$

so E_w carries a K_X -valued symplectic form. A straightforward calculation shows that α_w is the only Abel–Jacobi map which induces in this way a K_X -valued symplectic form on α_w^*M . We set $V_w = E_w \otimes \kappa^{-1}$ for any theta characteristic κ ; we will see shortly that the isomorphism class of V_w does not depend on this choice of κ . By the preceding discussion, we have an antisymmetric isomorphism $V_w \xrightarrow{\sim} V_w^*$.

This construction has an interesting corollary:

Proposition 27 *The bundle V_w is invariant under tensoring by all line bundles of order two in $\text{Pic}^0(X)$.*

Proof

Composing the isomorphisms $M^* \otimes O_J(\Theta_\kappa) \xrightarrow{\sim} M$ and $M \xrightarrow{\sim} M^* \otimes O_J(\Theta_{\kappa'})$ for each pair κ, κ' of distinct theta characteristics, we see that $M \cong M \otimes \eta$ for each two-torsion point η in J . By Birkenhake–Lange [9], Lemma 11.3.1, the map $\alpha_w^*: \text{Pic}^0(J) \rightarrow \text{Pic}^0(X)$ is an isomorphism, so E_w and V_w are also invariant under tensoring by line bundles of order two. \square

This proposition shows in particular that $E_w \otimes \kappa^{-1}$ and $E_w \otimes (\kappa')^{-1}$ are both isomorphic to V_w , as we needed.

In summary, we obtain a stable rank four symplectic bundle without a theta divisor for each odd theta characteristic of X .

9.3 The link with extensions

Here we make explicit the link between the bundles of Raynaud and Beauville just described and those constructed in section 6. The result, philosophically the only one possible, is that for each Weierstrass point w , the bundles V_w and W_w are isomorphic. This will follow easily from Lemma 28.

Lemma 28 *For any $x \in X$, the fibres of the subbundles $O_X(-x)$ and $O_X(-\iota x)$ of V_w coincide at the point w .*

Proof

Since $F(M) = R^0q_*(p^*M \otimes \mathbf{P})$ is isomorphic to $O_J(-2\Theta)$, by adjunction (Hartshorne [10], p. 110) we have a map $\Psi: q^*O_J(-2\Theta) \rightarrow p^*M \otimes \mathbf{P}$ over

$J \times J$. This has the property that for each $(L_1, L_2) \in J \times J$, the image of $\Psi|_{(L_1, L_2)}$ in

$$(p^*M \otimes \mathbf{P})|_{(L_1, L_2)} = (M \otimes L_2)|_{L_1}$$

is identified with that of the (unique) global section of $M \otimes L_2$ at L_1 .

We restrict Ψ to the subvariety $\{0\} \times J$ and denote Ψ_0 the induced map of vector bundles over J :

$$\begin{array}{ccc} (q^*O_J(-2\Theta))|_{\{0\} \times J} & \xrightarrow{\Psi|_{\{0\} \times J}} & (p^*M \otimes \mathbf{P})|_{\{0\} \times J} \\ \downarrow \parallel & & \downarrow \parallel \\ O_J(-2\Theta) & \xrightarrow{\Psi_0} & M|_0 \otimes (\mathbf{P}|_{\{0\} \times J}). \end{array}$$

Now recall that \mathbf{P} is trivial over $\{0\} \times J$, so Ψ_0 defines a section of $O_J(2\Theta)^{\oplus 4}$. Since every section of $O_J(2\Theta)$ is -1_J -invariant, we have a commutative diagram

$$\begin{array}{ccc} O_J(-2\Theta) & \xrightarrow{\Psi_0} & M|_0 \otimes (\mathbf{P}|_{\{0\} \times J}) \\ \downarrow \widetilde{(-1_J)} & & \downarrow \text{Id}_{M|_0} \otimes l \\ O_J(-2\Theta) & \xrightarrow{\Psi_0} & M|_0 \otimes (\mathbf{P}|_{\{0\} \times J}) \end{array}$$

where $\widetilde{(-1_J)}$ and l are suitable linearisations of $O_J(-2\Theta)$ and O_J respectively. This shows that the image of Ψ_0 at L coincides with that of L^{-1} under the identification of projective spaces $\mathbb{P}(M|_0 \otimes L) = \mathbb{P}(M|_0 \otimes L^{-1})$.

In particular the fibres of $O_X(x - w)$ and $O_X(x - w)^{-1} = O_X(\iota x - w)$ coincide in the fibre of the restricted bundle E_w , for each $x \in X$. This implies that the fibres of the subbundles $O_X(-x)$ and $O_X(-\iota x)$ of $E_w(-w) = V_w$ coincide in the fibre over $\alpha_w^{-1}(0) = w$, as required. \square

Theorem 29 *The bundle V_w associated to a Weierstrass point w is isomorphic to the extension W_w constructed in section 6.*

Proof

By Lemma 28, the fibres of the subbundles $O_X(-x)$ and $O_X(-\iota x)$ of V_w coincide at the point w . Since V_w is stable, they do not coincide anywhere else since then they would generate a rank two subbundle of degree at least zero. Thus V_w contains an subbundle of rank two which is an elementary transformation

$$0 \rightarrow \mathcal{O}_X(-x) \oplus \mathcal{O}_X(-\iota x) \rightarrow \mathcal{E} \rightarrow \mathbb{C}_w \rightarrow 0$$

Since E has degree -1 and belongs to V_w , it is stable. Then it is not hard to see that it is one of the bundles E_e that we considered in section 6. Moreover, it contains the subbundles $L^{-1}, K_X^{-1}L$ for some $L = \mathcal{O}_X(x) \in J^1$. Since we saw that $h^0(X, V_w(x)) = 1$ for all $x \in X$, Theorem 8 shows that E is isotropic in V_w . Thus V_w is a symplectic extension of E^* by E . Since by Lemma 14 there is only one such extension with no theta divisor, V_w must be isomorphic to W_w . \square

This establishes the link between the two constructions of the base points of $|\Xi|$.

10 Applications and future work

We give one immediate application of the finiteness of $\text{Bs}|\Xi|$.

Corollary 30 *The map $D: \mathcal{M}_2 \dashrightarrow |4\Theta|_+$ is surjective.*

Proof

Let G be an even 4Θ divisor. Choose nine hyperplanes Π_1, \dots, Π_9 in $|4\Theta|_+$ whose intersection consists of the point G . The inverse images $D^{-1}(\Pi_1), \dots, D^{-1}(\Pi_9)$ intersect in a subset S of dimension at least 1 in \mathcal{M}_2 since this variety is of dimension 10. Now S contains the base locus of $|\Xi|$. By Theorem 3, this is finite, so the map D is defined at most points of S . By construction, the image of any of these points is the divisor G . \square

We finish with a logically independent remark. It is not hard to see that the theta divisor of a *strictly semistable* bundle of degree zero, if it exists, must be reducible. The converse, however, is not true: there exist *stable* bundles with reducible theta divisors, as studied by Beauville [5]. In the case at hand, we can say:

Proposition 31 *Suppose X has genus two. Then there exist stable symplectic bundles in \mathcal{M}_2 with reducible theta divisors.*

Proof

Let V_1 and V_2 be mutually nonisomorphic stable bundles of rank two and trivial determinant over X with (irreducible) theta divisors $D_1, D_2 \in |2\Theta|$. Narasimhan and Ramanan showed in [25] that the theta map

$$D: \mathcal{SU}_X(2, \mathcal{O}_X) \rightarrow |2\Theta| = |2\Theta|_+$$

is an isomorphism. Hence $D_1 \neq D_2$. Let W be a strictly semistable bundle with $D(W) = D_1 + D_2$. With the list on p. 25 of [11], we consider all the

possibilities for W and, using [25], we find that the only one which can have a theta divisor of the form $D_1 + D_2$ is a direct sum of two stable bundles W_1 and W_2 of rank 2 and trivial determinant. We have

$$D(W_1 \oplus W_2) = D(W_1) + D(W_2) = D_1 + D_2$$

so, since J^1 is normal, $\{D(W_1), D(W_2)\} = \{D_1, D_2\}$.

But $\mathcal{SU}_X(2, \mathcal{O}_X)$ is isomorphic to $|2\Theta|$, so $W_1 \oplus W_2 \cong V_1 \oplus V_2$. Thus the fibre of D over $D(V_1 \oplus V_2) = D_1 + D_2$ contains at most one strictly semistable bundle. Since this fibre is of dimension at least one, there exist stable bundles with theta divisor $D_1 + D_2$. \square

Future work

By Criterion 4, one can construct bundles with orthogonal structure as extensions $0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0$ with classes in $H^1(X, \bigwedge^2 E)$. This space again carries an action of ι , so one might try to find orthogonal bundles without theta divisors with a construction analogous to that in section 6. One might also expect this construction to generalise to bundles of higher rank over hyperelliptic curves of higher genus.

Other natural things to look for include a description of the fibres of $D: \mathcal{M}_2 \dashrightarrow |4\Theta|_+$ and an explicit construction of the stable bundles with reducible theta divisors just mentioned. These questions will be studied in the future.

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A Proof of Proposition 20

Proposition 20 *The evaluation map $\text{ev}: O_X \otimes H^0(X, W \otimes K_X) \rightarrow W \otimes K_X$ is everywhere surjective to a stable subbundle of rank three and degree five in $W \otimes K_X$.*

Proof

The image of ev is a subsheaf \mathcal{F} of $W \otimes K_X$. We find first the rank of the corresponding vector bundle F , which is not necessarily a subbundle of $W \otimes K_X$. Firstly, since we have seen that for any $x \in X$ there is a section of $W \otimes K_X$ vanishing at x , the rank of F cannot be four.

We have $H^0(X, F) \cong H^0(X, W \otimes K_X)$, so $h^0(X, F) = 4$. Since $W \otimes K_X$ is stable, $\mu(F) < 2$. But no line bundle of degree one on X has four independent sections, so F is not of rank one. Suppose the rank is two. Then $\deg(F) \leq 3$. We now recall that it contains a subbundle $O_X(x)$ for every $x \in X$. This is because it has sections vanishing at each point of the curve, but has none vanishing at more than one point because such a section would generate a line subbundle of degree at least two in W . Thus we have a short exact sequence $0 \rightarrow O_X(x) \rightarrow F \rightarrow M \rightarrow 0$ for each $x \in X$, where M is a line bundle of degree at most two (depending on x). Then

$$h^0(X, F) \leq h^0(X, O_X(x)) + h^0(X, M) \leq 1 + 2 = 3,$$

a contradiction.

Thus F has rank three. By stability of $W \otimes K_X$, it has degree at most five. Suppose it is four or less. Let $x \in X$ be such that $\iota x \neq x$. As above, we have a short exact sequence $0 \rightarrow O_X(x) \rightarrow F \xrightarrow{q} G \rightarrow 0$ where G is of rank two and degree at most three.

By hypothesis, $h^0(X, G) \geq 3$. An upper bound for the degree d of a line subbundle of G can be calculated as follows: the inverse image in F of such a subbundle is a bundle of rank two and degree $d + 1$. By stability of W , we have $\frac{d+1}{2} < 2$, so $d \leq 2$.

Now let t be a section of F vanishing at $y \neq x$. The composed map $q \circ t$ generates a subbundle M of degree one or two in G , so we have an exact sequence

$$0 \rightarrow M \rightarrow G \rightarrow N \rightarrow 0$$

where N is a line bundle of degree at most $3 - \deg(M)$. Now

$$3 \leq h^0(X, G) \leq h^0(X, M) + h^0(X, N).$$

This shows that one of M and N is the canonical bundle and the other is effective (in particular, G has degree three). Suppose $N = K_X$; then $M = O_X(y)$. But we can do this for any y (with the same x), which implies

that $\det(G) = K_X(y)$ for all $y \in X$, which is clearly absurd. Hence $M = K_X$ and $N = O_X(p)$ for some $p \in X$. We write $p = p_x$ since p depends on x . Now $\det(F) = K_X(x + p_x) = K_X(x' + p_{x'})$ for generic $x' \in X$. Therefore $O_X(x + p_x) = O_X(x' + p_{x'})$ for all $x, x' \in X$, so $p_x = \iota x$ and G is an extension $0 \rightarrow K_X \rightarrow G \rightarrow O_X(\iota x) \rightarrow 0$.

Now by hypothesis, the unique section of $O_X(\iota x)$ lifts to G , so the extension class $\delta(G)$ belongs to the kernel of the induced map

$$H^1(X, \text{Hom}(O_X(\iota x), K_X)) \rightarrow H^1(X, \text{Hom}(O_X, K_X)),$$

which is identified with $H^0(X, O_X(\iota x))^* \rightarrow H^0(X, O_X)^*$ by Serre duality. But this is an isomorphism, so $G = K_X \oplus O_X(\iota x)$.

This means that F is an extension $0 \rightarrow O_X(x) \rightarrow F \rightarrow K_X \oplus O_X(\iota x) \rightarrow 0$ of class $\delta(F) \in H^1(X, \text{Hom}(K_X \oplus O_X(\iota x), O_X(x)))$, that is,

$$H^1(X, \text{Hom}(K_X, O_X(x))) \oplus H^1(X, \text{Hom}(O_X(\iota x), O_X(x))) = \mathbb{C}^3.$$

Now $h^0(X, K_X \oplus O_X(\iota x)) = 3$, so all sections of $K_X \oplus O_X(\iota x)$ lift to F by hypothesis. This means that the cup product map

$$\cdot \cup \delta(F): H^0(X, K_X \oplus O_X(\iota x)) \rightarrow H^1(X, O_X(x))$$

is zero. But by Lemma 6, the linear map

$$\begin{aligned} \cup: H^1(X, \text{Hom}(K_X \oplus O_X(\iota x), O_X(x))) \rightarrow \\ \text{Hom}(H^0(X, K_X \oplus O_X(\iota x)), H^1(X, O_X(x))) \end{aligned}$$

is dual to the multiplication map

$$H^0(X, K_X \oplus O_X(\iota x)) \otimes H^0(X, O_X(\iota x)) \rightarrow H^0(X, K_X(\iota x) \oplus O_X(2\iota x))$$

which is an isomorphism. Thus \cup is also an isomorphism, so $\delta(F) = 0$. But then K_X is a subbundle of F , which is not possible since $W \otimes K_X$ is stable.

The only possibility, then, is that F has degree five. To see that it is stable, note that by stability of $W \otimes K_X$, any line subbundle of F has slope at most 1, and any rank two subbundle has slope at most $3/2$. Both of these are less than $5/3 = \mu(F)$, so F is stable.

Lastly, let F' be the subbundle of $W \otimes K_X$ generated by the image of ev . This has degree at least five since \mathcal{F} has degree five. By stability of W , it is at most five. Hence $F' = F$ and ev is everywhere of rank three. This completes the proof. \square

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